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Integral Transforms of the Generalized Mittag-Leffler Function

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Abstract

This paper is devoted to study integral transforms of extended version of generalized Mittag-Leffler function introduced by Prajapati et al [9].

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1. Introduction

Shukla and Prajapati [2] have introduced the generalized form of the function

$$E_{\alpha,\beta}^{\gamma,q}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}z^n}{\Gamma(\alpha n+\beta)n!}, \tag{1.1}$$

and investigated its integral transforms and established its relationship with Laguerre polynomial, Fox $H-$function and Wright hypergeometric function. Srivastava and Tomovski [7] defined the integral operator

$$\int_0^x (x-t)^{\beta-1}E_{\alpha,\beta}^{\gamma,k}(w(x-t))dt, \tag{1.2}$$

And Prajapati et al [9] investigated its Laplace and Mellin transform. Prajapati and Shukla [8] decomposed the function defined by Shukla and Prajapati [2] in the form of truncated power series. Saxena et al [12] established the generalized form of the function which is the generalization of the function introduced by Srivastava and Tomovski [7] and is defined as

$$E_{\gamma,k}\left[\left(\alpha_j,\beta_j\right)_{1,m};z\right]=\sum_{n=0}^{\infty}\frac{(\gamma)_{kn}z^n}{\prod_{j=1}^{m}\Gamma(\alpha_j n+\beta_j)n!}. \tag{1.3}$$

They derived the integral representations of the function with the Euler integral of the first and second kind and the Laplace transform of the function.

Khan and Ahmed [11] have investigated the relation between Rieman Liouville fractional integrals and derivatives of the generalized Mittag-Leffler function and used the form of the function

$$E_{\alpha,\beta}^{\gamma,\delta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n z^n}{\Gamma(\alpha n+\beta)(\delta)_n}. \tag{1.4}$$

In continuation of this study, Salim and Faraj [14] introduced a new generalization of Mittag-Leffler function as

$$E_{\alpha,\beta,p}^{\gamma,\delta,q}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}z^n}{\Gamma(\alpha n+\beta)(\delta)_{pn}} \tag{1.5}$$

Faraj et al [1] have focused the behavior of the function defined by Salim and Faraj [14] and considered its association with Weyl fractional operators. Kiryakova [16] introduced multiindex Mittag-Leffler functions of order

$$\rho=\left(\sum_{i=1}^{n}\frac{1}{\rho_i}\right)^{-1} \text{ and type } \quad \sigma=\left(\frac{\rho_1}{\rho}\right)^{\rho/\rho_1}\cdots\left(\frac{\rho_m}{\rho}\right)^{\rho/\rho_m}. \tag{1.6}$$

Purohit et al [13] studied the fractional calculus of multiindex Mittag-Leffler functions defined by Kiryakova [16]. Gehlot [10] has used the $k-$Pochhammer symbol

$$(\xi)_{n,k}=\xi(\xi+k)(\xi+2k)\ldots(\xi+(n-1)k), \tag{1.7}$$

where $\xi\in C, k\in R$ and $n\in N$ and has defined more generalized form of the function known as generalized $k-$Mittag-Leffler function. Prajapati et al [9] have proposed an extension of generalized Mittag-Leffler function in the form of

$$E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}(z;s,r)=\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s z^{(pn+\rho-1)}}{\Gamma(\alpha(pn+\rho-1)+\beta)\left[(\lambda)_{\mu n}\right]^r(\rho)_{pn}} \tag{1.8}$$

The series of the function converges absolutely for $|z|<n^{1/p(\mathrm{Re}(\mu)r+\mathrm{Re}(\alpha)p-\mathrm{Re}(\delta)s+p)}$. Prajapati et al [9] have provided the integral representation of the function (1.9) in the form of Euler-Beta transform, Mellin-Barnes transform, Laplace transform, and Whittaker transform. We have established the relation of the function with other transforms like Henkal transforms and $K-$transform. Some definitions which will be further used in our findings are given below.

Wright generalized hypergeometric function:

$$ {}_p\Psi_q\left(\begin{matrix}(a_1,A_1),...,(a_p,A_p)\\(b_1,B_1),...,(b_q,B_q)\end{matrix};z\right)=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+A_i n)}{\prod_{j=1}^{q}\Gamma(b_j+B_j n)}\frac{z^n}{n!}. \tag{1.9}$$

(see Mathai et al ([4], pp.23)).
The transform defined by the following integral equation

$$R_v\{f(x);p\}=g(p,v)=\int_0^\infty (px)^{\frac{1}{2}}K_v(px)f(x)dx \tag{1.10}$$

is called the $K-$transform with p as a complex parameter and $K_v(px)$ is called Modified Bessel function of the third kind or Macdonald function.(see Mathai et al ([4], pp.53)).The Hankel transform of a function $f(x)$,denoted by $g(p,v)$ is defined as

$$g(p,v)=\int_0^\infty (px)^{\frac{1}{2}}J_v(px)f(x)dx,\ p>0 \tag{1.11}$$

where $J_v(px)$ is called Bessel–Maitland function or Maitland–Bessel function. (Mathai et al [4], (pp.22 &56)).

2. MAIN RESULTS

This section deals with the evaluation of integrals involving the generalized Mittag-Leffler function defined in (1.8). Integrals involving the product of Bessel function of first kind, Kelvin's function and Whittaker function with the generalized Mittag-Leffler function.

2.1 Theorem: (HANKEL TRANSFORM)

Let $\alpha,\beta,\gamma,\lambda,\sigma,b\in C$, $R(\alpha,\beta,\gamma,\lambda,\sigma,b)>0$ and $\delta,\mu,p>0$. Then the Henkel transform of the generalized Mittag- Leffler function defined in (1.8) is given by

$$\int_0^\infty z^{\sigma-1}J_v(az)E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(bz^w;s,r)=\frac{2^{\sigma-1}}{a^\sigma}\,{}_{s+3}\Psi_{r+3}\left[\begin{matrix}(\gamma,s)^s,(\frac{v+\sigma}{2},\frac{w}{2}),(1,1)\\(\lambda,\mu)^r,(\beta,\alpha),(\frac{2+v-\sigma}{2},\frac{-w}{2}),(\rho,p)\end{matrix};\left(b\left(\frac{2}{a}\right)^w\right)\right].$$

Proof:
Using the definition of (1.8) and (1.11), also due to the uniform convergence of the series, we get

$$\int_0^\infty z^{\sigma-1}J_v(az)E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(bz^w;s,r)=\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s b^{pn+\rho-1}}{\Gamma(\alpha(pn+\rho-1)+\beta)\left[(\lambda)_{\mu n}\right]^r(\rho)_{pn}}\int_0^\infty z^{\sigma-1}J_v(az)z^{w(pn+\rho-1)}dz$$

$$=\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s b^{pn+\rho-1}}{\Gamma(\alpha(pn+\rho-1)+\beta)\left[(\lambda)_{\mu n}\right]^r(\rho)_{pn}}\int_0^\infty z^{\sigma+w(pn+\rho-1)-1}J_v(az)dz$$

using the infinite integral of Bessel functions given by Mathai and Saxena [3]

$$\int_0^\infty x^{s-1} J_v(\alpha t)dt = \frac{2^{s-1}\alpha^{-s}\Gamma\left(\frac{v+s}{2}\right)}{\Gamma\left(1+\frac{v-s}{2}\right)},\ -R(v) < R(s) < \frac{3}{2}, \alpha > 0$$

Hence the integral will become,

$$\int_0^\infty z^{\sigma-1} J_v(az) E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(bz^w; s, r) = \sum_{n=0}^\infty \frac{\left[(\gamma)_{\delta n}\right]^s b^{pn+\rho-1}}{\Gamma(\alpha(pn+\rho-1)+\beta)\left[(\lambda)_{\mu n}\right]^r (\rho)_{pn}}$$

$$\text{x}\, 2^{\sigma+w(pn+\rho-1)-1} a^{-(\sigma+w(pn+\rho-1))} \frac{\Gamma\left(\frac{v+\sigma+w(pn+\rho-1)}{2}\right)}{\Gamma\left(\frac{2+v-\sigma-w(pn+\rho-1)}{2}\right)}$$

$$= \frac{2^{\sigma-1}}{a^\sigma} \sum_{n=0}^\infty \frac{\left[(\gamma)_{\delta n}\right]^s \left[b\left(\frac{2}{a}\right)^w\right]^{pn+\rho-1}}{\Gamma(\alpha(pn+\rho-1)+\beta)\left[(\lambda)_{\mu n}\right]^r (\rho)_{pn}} \frac{\Gamma\left(\frac{v+\sigma}{2}+\frac{w(pn+\rho-1)}{2}\right)}{\Gamma\left(\frac{2+v-\sigma}{2}-\frac{w(pn+\rho-1)}{2}\right)}$$

$$= \frac{2^{\sigma-1}}{a^\sigma} {}_{s+3}\Psi_{r+3}\left[\begin{matrix} (\gamma,s)^s, (\frac{v+\sigma}{2}, \frac{w}{2}), (1,1) \\ (\lambda,\mu)^r, (\beta,\alpha), (\frac{2+v-\sigma}{2}, \frac{-w}{2}), (\rho, p) \end{matrix} ; \left(b\left(\frac{2}{a}\right)^w\right)\right].$$

2.2 Theorem ($K-$TRANSFORM)

$$\int_0^\infty z^{\sigma-1} K_v(az) E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(bz^w) = \frac{2^{\sigma-2}}{a^\sigma} {}_{s+3}\Psi_{r+2}\left(\begin{matrix} (\gamma,\delta)^s, (1,1), (\frac{\sigma\pm v}{2}, \frac{w}{2}) \\ (\beta,\alpha), (\lambda,\mu)^r, (\rho, p) \end{matrix} ; b\left[\frac{2}{a}\right]^w\right)$$

Proof:

If we apply the definition of (1.8)

$$\int_0^\infty z^{\sigma-1} K_v(az) E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(bz^w) = \sum_{n=0}^\infty \frac{\left[(\gamma)_{\delta n}\right]^s b^{pn+\rho-1}}{\Gamma(\alpha(pn+\rho-1)+\beta)\left[(\lambda)_{\mu n}\right]^r (\rho)_{pn}} \int_0^\infty z^{\sigma+w(pn+\rho-1)-1} K_v(az)dz$$

using the integral of Mathai and Saxena [3]

$$\int_0^\infty x^{\rho-1} K_v(ax)dx = 2^{\rho-2} a^{-\rho} \Gamma\left(\frac{\rho\pm v}{2}\right).$$

$$\int_0^\infty z^{\sigma-1}K_v(az)E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(bz^w)=\sum_{n=0}^\infty\frac{\left[(\gamma)_{\delta n}\right]^s b^{pn+\rho-1}}{\Gamma(\alpha(pn+\rho-1)+\beta)\left[(\lambda)_{\mu n}\right]^r(\rho)_{pn}}$$

$$\text{x}\qquad 2^{\sigma+w(pn+\rho-1)-2}a^{-[\sigma+w(pn+\rho-1)]}\Gamma\left[\frac{\sigma+w(pn+\rho-1)\pm v}{2}\right]$$

$$=\frac{2^{\sigma-2}}{a^\sigma}\sum_{n=0}^\infty\frac{(\gamma+\delta n)^s\left[b\left(\frac{2}{a}\right)^w\right]^{pn+\rho-1}}{\Gamma(\alpha(pn+\rho-1)+\beta)\left[(\lambda+\mu n)\right]^r\Gamma(\rho+pn)}\Gamma\left[\frac{\sigma\pm v}{2}+\frac{w(pn+\rho-1)}{2}\right]$$

$$=\frac{2^{\sigma-2}}{a^\sigma}\,{}_{s+3}\Psi_{r+2}\left(\begin{matrix}(\gamma,\delta)^s,(1,1),(\frac{\sigma\pm v}{2},\frac{w}{2})\\(\beta,\alpha),(\lambda,\mu)^r,(\rho,p)\end{matrix};b\left[\frac{2}{a}\right]^w\right).$$

2.3 Theorem ($K-$TRANSFORM)

$$\int_0^\infty z^{\sigma-1}e^{-az}K_v(az)E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(bz^w)dz=\frac{\pi^{\frac{1}{2}}}{(2a)^\sigma}\,{}_{s+3}\Psi_{r+3}\left(\begin{matrix}(\gamma,\delta)^s,(\sigma\pm v,w),(1,1)\\(\beta,\alpha),(\lambda,\mu)^r,(\rho,p),(\sigma+\frac{1}{2},w)\end{matrix};\frac{b}{(2a)^w}\right).$$

Proof:

$$\int_0^\infty z^{\sigma-1}e^{-az}K_v(az)E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(bz^w)dz=\sum_{n=0}^\infty\frac{\left[(\gamma)_{\delta n}\right]^s b^{pn+\rho-1}}{\Gamma(\alpha(pn+\rho-1)+\beta)\left[(\lambda)_{\mu n}\right]^r(\rho)_{pn}}\int_0^\infty z^{\sigma+w(pn+\rho-1)-1}e^{-az}K_v(az)dz$$

using the integral of Mathai and Saxena [3]

$$\int_0^\infty z^{\sigma-1}e^{-az}K_v(az)E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(bz^w)dz=\sum_{n=0}^\infty\frac{\left[(\gamma)_{\delta n}\right]^s b^{pn+\rho-1}}{\Gamma(\alpha(pn+\rho-1)+\beta)\left[(\lambda)_{\mu n}\right]^r(\rho)_{pn}}$$

$$\text{x}\quad\frac{\pi^{\frac{1}{2}}\Gamma(\sigma+w(pn+\rho-1)\pm v)}{(2a)^{\sigma+w(pn+\rho-1)}\Gamma(\sigma+w(pn+\rho-1)+\frac{1}{2})}$$

$$=\frac{\pi^{\frac{1}{2}}}{(2a)^\sigma}\sum_{n=0}^\infty\frac{\left(\Gamma(\gamma+\delta n)\right)^s\Gamma(\sigma\pm v+w(pn+\rho-1))\Gamma(1+k)}{k!\Gamma(\sigma+\frac{1}{2}+w(pn+\rho-1))\Gamma(\beta+\alpha(pn+\rho-1))\left[\Gamma(\lambda+\mu n)\right]^r\Gamma(\rho+pn)}$$

$$=\frac{\pi^{\frac{1}{2}}}{(2a)^\sigma}\,{}_{s+3}\Psi_{r+3}\left(\begin{matrix}(\gamma,\delta)^s,(\sigma\pm v,w),(1,1)\\(\beta,\alpha),(\lambda,\mu)^r,(\rho,p),(\sigma+\frac{1}{2},w)\end{matrix};\frac{b}{(2a)^w}\right).$$

3. INTEGRALS INVOLVING THE PRODUCT OF WHITTAKER FUNCTION AND MITTAG-LEFFLER FUNCTION

3.1 Theorem (VARMA TRANSFORM):

$$\int_0^\infty t^{v-1} e^{\frac{1}{2}\sigma t} W_{\eta,\chi}(\sigma t) E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}(wt^{\xi})dt = \frac{1}{\sigma^v}\left(\frac{w}{\sigma^{\xi}}\right)^{\rho-1} \frac{[\Gamma(\lambda)]^r \Gamma(\rho)}{[\Gamma(\gamma)]^s \Gamma\left(\frac{1}{2}-\lambda \pm \mu\right)}$$

$$\text{x} \quad {}_{s+4}\Psi_{r+2}\left[\begin{matrix}(\gamma,\delta)^s, (\frac{1}{2}\pm\chi+v+\xi\rho-\xi, \xi p), (\xi-\lambda-v-\xi\rho, -\xi p), (1,1) \\ (\alpha\rho-\alpha+\beta, \alpha p), (\lambda,\mu)^r, (\rho, p)\end{matrix}; \left(\frac{w}{\sigma^{\xi}}\right)\right].$$

Proof:

Substituting $\sigma t = x, \sigma dt = dx$ as $t \to 0, x \to 0$ and $t \to \infty, x \to \infty$,

$$\int_0^\infty t^{v-1} e^{\frac{1}{2}\sigma t} W_{\eta,\chi}(\sigma t) E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}(wt^{\xi})dt = \int_0^\infty \left(\frac{x}{\sigma}\right)^{v-1} e^{\frac{1}{2}x} W_{\eta,\chi}(x) \sum_{n=0}^{\infty} \frac{\left[(\gamma)_{\delta n}\right]^s \left(w\left(\frac{x}{\sigma}\right)^{\xi}\right)^{pn+\rho-1}}{\Gamma(\alpha(pn+\rho-1)+\beta)\left[(\lambda)_{\mu n}\right]^r (\rho)_{pn}} \frac{dx}{\sigma}$$

$$= \frac{1}{\sigma^v} \sum_{n=0}^{\infty} \frac{\left[(\gamma)_{\delta n}\right]^s \left(\frac{w}{\sigma^{\xi}}\right)^{pn+\rho-1}}{\Gamma(\alpha(pn+\rho-1)+\beta)\left[(\lambda)_{\mu n}\right]^r (\rho)_{pn}} \int_0^\infty x^{v+\xi(pn+\rho-1)-1} e^{\frac{1}{2}x} W_{\eta,\chi}(x)dx$$

Since the known integral of Mathai and Saxena [3]

$$\int_0^\infty t^{v-1} e^{\frac{1}{2}t} W_{\lambda,\mu}(t)dt = \frac{\Gamma\left(\frac{1}{2}\pm\mu+v\right)\Gamma(-\lambda-v)}{\Gamma\left(\frac{1}{2}\pm\mu-\lambda\right)}.$$

Thus the result will become

$$\int_0^\infty t^{v-1}e^{\frac{1}{2}\sigma t}W_{\eta,\chi}(\sigma t)E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(wt^{\xi})dt=\frac{1}{\sigma^v}\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s\left(\frac{w}{\sigma^{\xi}}\right)^{pn+\rho-1}}{\Gamma(\alpha(pn+\rho-1)+\beta)\left[(\lambda)_{\mu n}\right]^r(\rho)_{pn}}$$

$$\text{x}\ \frac{\Gamma\left(\frac{1}{2}\pm\chi+v+\xi(pn+\rho-1)\right)\Gamma\left(-\lambda-v-\xi(pn+\rho-1)\right)}{\Gamma\left(\frac{1}{2}-\lambda\pm\mu\right)}$$

$$=\frac{1}{\sigma^v}\left(\frac{w}{\sigma^{\xi}}\right)^{\rho-1}\frac{\left[\Gamma(\lambda)\right]^r\Gamma(\rho)}{\left[\Gamma(\gamma)\right]^s\Gamma\left(\frac{1}{2}-\lambda\pm\mu\right)}$$

$$\text{x}\ \sum_{n=0}^{\infty}\frac{\left[\Gamma(\gamma+\delta n)\right]^s\Gamma(\rho)\Gamma\left(\frac{1}{2}\pm\chi+v+\xi\rho-\xi+\xi pn\right)\Gamma\left(\xi-\lambda-v+\xi\rho-\xi pn)\right)}{\Gamma(\alpha\rho-\alpha+\beta+\alpha pn)\left[(\lambda+\mu n)\right]^r\Gamma(\rho+pn)}\left(\frac{w}{\sigma^{\xi}}\right)^{pn}$$

$$=\frac{1}{\sigma^v}\left(\frac{w}{\sigma^{\xi}}\right)^{\rho-1}\frac{\left[\Gamma(\lambda)\right]^r\Gamma(\rho)}{\left[\Gamma(\gamma)\right]^s\Gamma\left(\frac{1}{2}-\lambda\pm\mu\right)}\,{}_{s+4}\Psi_{r+2}\left[\begin{matrix}(\gamma,\delta)^s,(\frac{1}{2}\pm\chi+v+\xi\rho-\xi,\xi p),(\xi-\lambda-v-\xi\rho,-\xi p),(1,1)\\(\alpha\rho-\alpha+\beta,\alpha p),(\lambda,\mu)^r,(\rho,p)\end{matrix};\left(\frac{w}{\sigma^{\xi}}\right)^p\right].$$

3.2 Theorem:

$$\int_0^\infty t^{v-1}e^{\frac{1}{2}\sigma t}M_{\eta,m}(\sigma t)E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(wt^{\xi})dt=\frac{1}{\sigma^v}\left(\frac{w}{\sigma^{\xi}}\right)^{\rho-1}\frac{\Gamma(2m+1)\Gamma(\rho)\left[\Gamma(\lambda)\right]^r}{\left[\Gamma(\gamma)\right]^s\Gamma\left(m+\frac{1}{2}+\eta\right)}$$

$$\text{x}\ {}_{s+3}\Psi_{r+3}\left[\begin{matrix}(\gamma,\delta)^s,(m+v+\frac{1}{2}+\xi\rho-\xi,\xi p),(\xi+\eta-v-\xi\rho,-\xi p),(1,1)\\(\alpha\rho-\alpha+\beta,\alpha p),(\lambda,\mu)^r,(\rho,p),(m-v+\frac{1}{2}-\xi\rho+\xi,-\xi p)\end{matrix};\left(\frac{w}{\sigma^{\xi}}\right)^p\right].$$

Proof:

Substituting $\sigma t=x,\sigma dt=dx$ as $t\to 0,x\to 0$ and $t\to\infty,x\to\infty$,

$$\int_0^\infty t^{v-1}e^{-\frac{1}{2}\sigma t}M_{\eta,m}(\sigma t)E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(wt^{\xi})dt=\int_0^\infty\left(\frac{x}{\sigma}\right)^{v-1}e^{-\frac{1}{2}x}M_{\eta,m}(x)\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s\left(w\left(\frac{x}{\sigma}\right)^{\xi}\right)^{pn+\rho-1}}{\Gamma(\alpha(pn+\rho-1)+\beta)\left[(\lambda)_{\mu n}\right]^r(\rho)_{pn}}\frac{dx}{\sigma}$$

Due to the uniform convergence the order of summation and integral can be changed

$$= \frac{1}{\sigma^{v}} \sum_{n=0}^{\infty} \frac{\left[(\gamma)_{\delta n}\right]^{s} \left(\frac{w}{\sigma^{\xi}}\right)^{pn+\rho-1}}{\Gamma(\alpha(pn+\rho-1)+\beta)\left[(\lambda)_{\mu n}\right]^{r} (\rho)_{pn}} \int_{0}^{\infty} x^{v+\xi(pn+\rho-1)-1} e^{-\frac{1}{2}x} M_{\eta,m}(x)dx$$

Since the known integral of Mathai and Saxena [3]

$$\int_{0}^{\infty} t^{v-1} e^{-\frac{1}{2}t} M_{\lambda,m}(t)dt = \frac{\Gamma(2m+1)\Gamma(m+v+\frac{1}{2})\Gamma(\mu-v)}{\Gamma\left(m-v+\frac{1}{2}\right)\Gamma\left(m+\mu+\frac{1}{2}\right)}.$$

Thus the result will become

$$\int_{0}^{\infty} t^{v-1} e^{-\frac{1}{2}\sigma t} M_{\eta,m}(\sigma t) E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(wt^{\xi})dt = \frac{1}{\sigma^{v}} \sum_{n=0}^{\infty} \frac{\left[(\gamma)_{\delta n}\right]^{s} \left(\frac{w}{\sigma^{\xi}}\right)^{pn+\rho-1}}{\Gamma(\alpha(pn+\rho-1)+\beta)\left[(\lambda)_{\mu n}\right]^{r} (\rho)_{pn}}$$

$$\text{x} \; \frac{\Gamma(2m+1)\Gamma\left(m+v+\frac{1}{2}+\xi(pn+\rho-1)\right)\Gamma\left(\eta-v-\xi(pn+\rho-1)\right)}{\Gamma\left(m-v-\xi(pn+\rho-1)+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}+\xi\right)}$$

$$= \frac{1}{\sigma^{v}} \left(\frac{w}{\sigma^{\xi}}\right)^{\rho-1} \frac{\Gamma(2m+1)\left[\Gamma(\lambda)\right]^{r} \Gamma(\rho)}{\left[\Gamma(\gamma)\right]^{s} \Gamma\left(m+\frac{1}{2}+\eta\right)}$$

$$\text{x} \sum_{n=0}^{\infty} \frac{\left[\Gamma(\gamma+\delta n)\right]^{s} \Gamma\left(m+v+\frac{1}{2}+\xi\rho-\xi+\xi pn\right)\Gamma\left(\eta-v-\xi\rho+\xi-\xi pn\right)}{\Gamma(\alpha\rho-\alpha+\beta+\alpha pn)\left[(\lambda+\mu n)\right]^{r} \Gamma(\rho+pn)\Gamma\left(m-v+\frac{1}{2}-\xi\rho+\xi+\xi pn\right)} \left(\frac{w}{\sigma^{\xi}}\right)^{pn}$$

$$= \frac{1}{\sigma^{v}} \left(\frac{w}{\sigma^{\xi}}\right)^{\rho-1} \frac{\Gamma(2m+1)\left[\Gamma(\lambda)\right]^{r} \Gamma(\rho)}{\left[\Gamma(\gamma)\right]^{s} \Gamma\left(m+\frac{1}{2}+\eta\right)} {}_{s+3}\Psi_{r+3}\left[\begin{matrix} (\gamma,\delta)^{s}, (m+v+\frac{1}{2}+\xi\rho-\xi, \xi p), (\eta-v-\xi\rho+\xi, -\xi p), (1,1) \\ (\alpha\rho-\alpha+\beta, \alpha p), (\lambda,\mu)^{r}, (\rho, p), (m-v+\frac{1}{2}-\xi\rho+\xi, -\xi p) \end{matrix} ; \left(\frac{w}{\sigma^{\xi}}\right)^{p}\right].$$

3.3 Theorem:

$$\int_{0}^{\infty} t^{v-1} W_{\eta,\chi}(\sigma t) W_{-\eta,\chi}(\sigma t) E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(wt^{\xi})dt = \frac{1}{2\sigma^{v}} \left(\frac{w}{\sigma^{\xi}}\right)^{\rho-1} \frac{\left[\Gamma(\lambda)\right]^{r} \Gamma(\rho)}{\left[\Gamma(\gamma)\right]^{s}}$$

$$\text{x} \; {}_{s+4}\Psi_{r+4}\left[\begin{matrix} (\gamma,\delta)^{s}, (v+\xi\rho-\xi+1, \xi p), \left(\frac{v-\xi+\xi\rho\pm 2\chi}{2}, \frac{\xi p}{2}\right), (1,1) \\ (\alpha\rho-\alpha+\beta, \alpha p), (\lambda,\mu)^{r}, (\rho, p), \left(\frac{2+v-\xi+\xi\rho\pm 2\lambda}{2}, \frac{\xi p}{2}\right) \end{matrix} ; \left(\frac{w}{\sigma^{\xi}}\right)^{p}\right].$$

Proof:

Substituting $\sigma t = x, \sigma dt = dx$ as $t \to 0, x \to 0$ and $t \to \infty, x \to \infty,$

$$\int_0^\infty t^{v-1} W_{\eta,\chi}(\sigma t) W_{-\eta,\chi}(\sigma t) E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(wt^{\xi}) dt = \int_0^\infty \left(\frac{x}{\sigma}\right)^{v-1} W_{\eta,\chi}(x) W_{-\eta,\chi}(x) \sum_{n=0}^{\infty} \frac{\left[(\gamma)_{\delta n}\right]^s \left(w\left(\frac{x}{\sigma}\right)^{\xi}\right)^{pn+\rho-1}}{\Gamma(\alpha(pn+\rho-1)+\beta)\left[(\lambda)_{\mu n}\right]^r (\rho)_{pn}} \frac{dx}{\sigma}$$

$$= \frac{w^{\rho-1}}{\sigma^{v+\xi(\rho-1)}} \sum_{n=0}^{\infty} \frac{\left[(\gamma)_{\delta n}\right]^s \left(\frac{w}{\sigma^{\xi}}\right)^{pn}}{\Gamma(\alpha(pn+\rho-1)+\beta)\left[(\lambda)_{\mu n}\right]^r (\rho)_{pn}} \int_0^\infty x^{v+\xi(pn+\rho-1)-1} W_{\eta,\chi}(x) W_{-\eta,\chi}(x) dx$$

Since the known integral of Mathai and Saxena [3]

$$\int_0^\infty t^{v-1} W_{\lambda,\mu}(t) W_{-\lambda,\mu}(t) dt = \frac{\Gamma\left(\frac{v+1}{2} \pm \mu\right)\Gamma(v+1)}{2\Gamma\left(1+\frac{v}{2} \pm \lambda\right)}.$$

Thus the result will become

$$\int_0^\infty t^{v-1} W_{\eta,\chi}(\sigma t) W_{-\eta,\chi}(\sigma t) E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(wt^{\xi}) dt =$$

$$\frac{w^{\rho-1}}{\sigma^{v+\xi(\rho-1)}} \frac{\left[\Gamma(\lambda)\right]^r \Gamma(\rho)}{\left[\Gamma(\gamma)\right]^s} \sum_{n=0}^{\infty} \frac{\left[\Gamma(\gamma+\delta n)\right]^s}{\Gamma(\alpha(pn+\rho-1)+\beta)\left[\Gamma(\lambda+\mu n)\right]^r \Gamma(\rho+pn)}$$

$$\text{x} \quad \frac{\Gamma\left(v+\xi(pn+\rho-1)+1\right)\Gamma\left(\frac{v+\xi(pn+\rho-1)+1}{2} \pm \chi\right)}{2\Gamma\left(1+\frac{v+\xi(pn+\rho-1)}{2} \pm \lambda\right)} \left(\frac{w}{\sigma^{\xi}}\right)^{pn}$$

$$= \frac{1}{2\sigma^{v}} \left(\frac{w}{\sigma^{\xi}}\right)^{\rho-1} \frac{\left[\Gamma(\lambda)\right]^r \Gamma(\rho)}{\left[\Gamma(\gamma)\right]^s} {}_{s+4}\Psi_{r+4}\left[\begin{matrix} (\gamma,\delta)^s, (v+\xi\rho-\xi+1, \xi p), \left(\frac{v-\xi+\xi\rho \pm 2\chi}{2}, \frac{\xi p}{2}\right), (1,1) \\ (\alpha\rho-\alpha+\beta, \alpha p), (\lambda,\mu)^r, (\rho, p), \left(\frac{2+v-\xi+\xi\rho \pm 2\lambda}{2}, \frac{\xi p}{2}\right) \end{matrix}; \left(\frac{w}{\sigma^{\xi}}\right)^{p}\right].$$

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