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Parameter Estimation for the Bivariate Lomax Distribution Based on Censored Samples

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Abstract

In this paper, we estimate the parameters of the bivariate Lomax distribution of Marshall-Olkin bsed on right censored sample. The advantage of using the EM algorithm is that the observed data vector is viewed as being incomplete but regarded as an observable function of complete data. Then the EM algorithm exploits the reduced complexity of maximum likelihood estimation for complete data. We also derive the standard deviations of the estimates for this bivariate Lomax distribution. A simulation study is conducted to compare the estimated values with the true values.

Keywords: Bivariate Lomax Distribution**,** censored sample, EM algorithm, missing data, simulation study

1. Introduction

The Lomax (1954), or Pareto II, distribution has been quite widely applied in a variety of contexts. Although introduced originally for modelling business failure data, the Lomax distribution has been used for reliability modelling and life testing (*e.g.*, Hassan and Al-Ghamdi, 2009), and applied to income and wealth distribution data (Harris, 1968; Atkinson and Harrison, 1978), firm size (Corbelini *et al.*, 2007) and queuing problems. It has also found application in the biological sciences and even for modelling the distribution of the sizes of computer files on servers (Holland *et al*., 2006). Some authors, such as Bryson (1974), have suggested the use of this distribution as an alternative to the exponential distribution when the data are heavy-tailed.

In this paper, maximum likelihood estimates (MLEs) of the parameters of the bivariate Lomax distribution (BVL) of Marshall-Olkin (1967) are obtained based on censored samples . The censoring time (T) is assumed to be independent of the life times (X, Y) of the two components.

The considered situation occurs for example in medical studies of paired organs like kidneys, eyes, lungs, or any other paired organs of an individual as a two components system which works under interdependency circumstances. Failure of an individual may censor failure of either one of the paired organ or both. This scheme of censoring is right censoring.

There are similar situation in engineering science whenever sub-systems are considered having two components with life times (X, Y) being independent of the life time (T) of the entire system. However, failure of the main system may censor failure of either one component or both. Hanagal (1992a, 1992b) derived maximum likelihood estimators of the parameters for the case of univariate right censoring.

Censoring may also occur in other ways. Patients may be lost to follow up during the study, the patient may decide to move elsewhere therefore the experimenter may not follow him or her again, or the patients may become non-cooperative which is due to some bad side effects of the therapy. Such cases are called withdrawal from the study. A patient with censored data contributes valuable information and should therefore not be omitted from the analysis.

Because of lack of data of real processes, the data in this study are generated from the BVL of Marshall-Olkin (1967) using Matlab software. Subsequently the EM algorithm is used to estimate the parameters.

2. Marshall-Olkin Bivariate Lomax Distribution

Let X be a random variable with the following CDF as follows;

$$F(x) = 1 - \left(1 + \frac{x}{k}\right)^{-\alpha} \qquad , x > 0\,, \alpha > 0, k > 0 \qquad (1)$$

The distribution of this form is said to be a Lomax distribution with parameters $\alpha \; and \; k$, will be denoted by $Lom(\alpha, k)$. The PDF and the hazard function of Lomax distribution with parameters (α, k) will be

$$f_L(x; \alpha, k) = \frac{\alpha}{k}\left(1 + \frac{x}{k}\right)^{-\alpha-1} \qquad , x > 0 \qquad (2)$$

Suppose U_1, U_2 and U_3 are three independent random variables such that $U_i \sim Lom(\alpha_i, k)$ for i = 1, 2 and 3 it is assumed that $\alpha_1, \alpha_2, \alpha_3, k > 0$. Define

$X = \min(u_1, u_3)$ and $Y = \min(u_2, u_3)$.

Then we say that the bivariate vector (X, Y) has a bivariate Lomax distribution Marshall-Olkin (1967).

The joint survival function of *X* and *Y* is defined as

$$\bar{F}_{X,Y}(x,y) = \left(1+\frac{x}{k}\right)^{-\alpha_1}\left(1+\frac{y}{k}\right)^{-\alpha_2}\left(1+\frac{\max(x,y).}{k}\right)^{-\alpha_3}$$
$$= \bar{F}_L(x;\alpha_1,k)\bar{F}_L(y;\alpha_2,k)\bar{F}_L(\max(x,y).;\alpha_3,k) \quad (3)$$

$$\bar{F}_{X,Y}(x,y) = \begin{cases} \bar{F}_1(x,y) & 0<x<y<\infty \\ \bar{F}_2(x,y) & 0<y<x<\infty \\ \bar{F}_0(x) & 0<x=y<\infty \end{cases} \quad (4)$$

where

$$\bar{F}_1(x,y) = \left(1+\frac{y}{k}\right)^{-(\alpha_2+\alpha_3)}\left(1+\frac{x}{k}\right)^{-\alpha_1}$$
$$= \bar{F}(y;\alpha_2+\alpha_3,k)\bar{F}(x;\alpha_1,k)$$
$$\bar{F}_2(x,y) = \left(1+\frac{x}{k}\right)^{-(\alpha_1+\alpha_3)}\left(1+\frac{y}{k}\right)^{-\alpha_2}$$
$$= \bar{F}(x;\alpha_1+\alpha_3,k)\bar{F}(y;\alpha_2,k)$$
$$\bar{F}_0(x) = \left(1+\frac{x}{k}\right)^{-\alpha} = \bar{F}(x;\alpha,k)$$

where $\alpha = \alpha_1+\alpha_2+\alpha_3$

The joint probability density function $f_{X,Y}(x,y)$ of X and Y takes the form

$$f_{X,Y}(x,y)\begin{cases} f_1(x,y) & if \quad x<y \\ f_2(x,y) & if \quad y<x \\ f_0(x) & if \quad x=y \end{cases} \quad (5)$$

where

$$f_1(x,y) = \frac{(\alpha_2+\alpha_3)\alpha_1}{k^2}\left(1+\frac{y}{k}\right)^{-(\alpha_2+\alpha_3)-1}\left(1+\frac{x}{k}\right)^{-\alpha_1-1}$$
$$f_2(x,y) = \frac{(\alpha_1+\alpha_3)\alpha_2}{k^2}\left(1+\frac{x}{k}\right)^{-(\alpha_1+\alpha_3)-1}\left(1+\frac{y}{k}\right)^{-\alpha_2-1}$$

and

$$f_0(x) = \frac{\alpha_3}{k}\left(1+\frac{x}{k}\right)^{-\alpha-1}$$

The density functions of $X|\{(x,y)|x>y\}, Y|\{(x,y)|y>x\}$ and Z=min (X,Y) are given as follows:

$$f_{X|(x,y)}(x) = \frac{(\alpha_1+\alpha_3)}{k}\left(1+\frac{x}{k}\right)^{-(\alpha_1+\alpha_3)-1}\left(1+\frac{y}{k}\right)^{\alpha_1+\alpha_3}, \quad y<x$$
$$f_{Y|(x,y)}(y) = \frac{(\alpha_2+\alpha_3)}{k}\left(1+\frac{y}{k}\right)^{-(\alpha_2+\alpha_3)-1}\left(1+\frac{x}{k}\right)^{\alpha_2+\alpha_3}, \quad x<y \quad (6)$$
$$f_Z(z) = \frac{(\alpha)}{k}\left(1+\frac{z}{k}\right)^{-(\alpha)-1}, \quad Z = \min(x,y)$$

This paper aims at deriving an estimation method for the parameters of a bivariate Lomax distribution of Marshall-Olkin by the EM algorithm.

In Section 2, the EM algorithm is presented, while in Section 3, the parameters are estimated by applying the EM algorithm. Finally in Section 4, we present the results of a simulation study.

3. The EM Algorithm

The expectation-maximization (EM) algorithm was introduced by Dempster et al (1977). The algorithm is an iterative procedure for finding the maximum likelihood estimates for incomplete, missing, unobserved or censored data. Because it is easy to implement, the impact of the EM algorithm has been far reaching, not only as computational tool but also as a way of solving difficult statistical problems.

The basic idea behind the method is to transform a set of incomplete data into a complete data problem for which the required maximization is computationally more tractable and numerically stable. Each iteration increases the likelihood, which finally converges almost always to a local maximum.

One can view the complete data set x as consisting of the vectors(t, t^*), where t is the observed, i.e., incomplete data, and t^* is the missing data.

3.1. The Iterations

The objective is to draw inferences about the parameter vectors $\underline{\alpha} \in \Omega$. We will use $L(\underline{\alpha}|t)$ to denote the likelihood function where t is the vector of observed data. Let t^* represent the vector of missing data. Starting with a guessed value for the parameter $\underline{\alpha}$, carry out the following iterations:

1. Replace the missing data t^* by their expectation given the guessed value of the parameter vector and the observed data. Let this conditional expectation be denoted by $\widetilde{t^*}$.
2. Maximize$L(\underline{\alpha}, t^*|t)$ with respect to $\underline{\alpha}$ replacing the missing data t^* by their expected values. This is equivalent to maximizing $L(\underline{\alpha}, E\,[T^*|t])$.
3. Re-estimate the missing values t^* using their conditional expectation based on the updated $\underline{\alpha}$.
4. Re-estimate $\underline{\alpha}$ and continue until the difference between a new iterated value and the previous iterated value is less than 0.00001.

3.2. Execution of the Algorithm

Consider the logarithm of the likelihood $\log L(\underline{\alpha}, t^*|t)$. Let the conditional expectation of $\log L(\underline{\alpha}, t^*|t)$ with respect to $t^*|(\underline{\alpha}^{(k)}, t)$ at k^{th} step be noted by $Q(\underline{\alpha}|\underline{\alpha}^{(k)})$, where $\underline{\alpha}^{(k)}$ is the current guess of $\underline{\alpha}$.Then the EM-steps are as follows:

1. E-step: Calculate $Q(\underline{\alpha}|\underline{\alpha}^{(k)})$, that is, that is, the expectation of the log-likelihood with respect to the conditional distribution of the missing data, given the observed data and the current guess of $\underline{\alpha}$.
2. M-step: Maximize $Q(\underline{\alpha}|\underline{\alpha}^{(k)})$with respect to $\underline{\alpha}$ and set the result equal to $\underline{\alpha}^{(k+1)}$, the new value of the parameter vector. Since $\underline{\alpha}^{(k+1)}$

maximizes $Q(\underline{\alpha}|\underline{\alpha}^{(k)})$the M- Step results in $Q(\underline{\alpha}^{(k+1)}|\underline{\alpha}^{(k)}) \geq Q(\underline{\alpha}|\underline{\alpha}^{(k)})$ for all $\underline{\alpha} \in \Omega$, implying that $\underline{\alpha}^{(k+1)}$ is a solution to equation, $\frac{\partial Q(\underline{\alpha}|\underline{\alpha}^{(k)})}{\partial \underline{\alpha}} = 0$.

The two steps are repeated iteratively until the difference between two successive iterations is less than 0.00001. This iterative procedure leads to a monotonic increase of log $L(\underline{\alpha}, E[T^*|t])$:

$$logL(\underline{\alpha}^{(k+1)}, E[T^*|t]) \geq logL(\underline{\alpha}, E[T^*|t]) \qquad \text{for } k = 1,2,\ldots \tag{7}$$

Since the likelihood increases in each step, the EM algorithm converges generally to a local maximum. When there is no closed form solution of the M-step, a numerical algorithm as, for example, the Newton-Raphson procedure, may be used for iteratively computing $\underline{\alpha}^{(k)}$. In fact, in this paper the Newton-Raphson procedure is used to obtain maximum likelihood estimates of $\underline{\alpha}$ at the *(k +1)* th iteration as follows:

$$\underline{\alpha}^{(k+1)} = \underline{\alpha}^{(k)} - \left(\frac{\partial^2 Q(\underline{\alpha}|\underline{\alpha}^{(k)})}{\partial \underline{\alpha} \partial \underline{\acute{\alpha}}}\right)^{-1}_{\underline{\alpha}=\underline{\alpha}^{(k)}} \left(\frac{\partial Q(\underline{\alpha}|\underline{\alpha}^{(k)})}{\partial \underline{\alpha}}\right)_{\underline{\alpha}=\underline{\alpha}^{(k)}} \tag{8}$$

The iterative procedure is carried out until the difference $\underline{\alpha}^{(k+1)} - \underline{\alpha}^{(k)} <$ 0.00001.

3. Parameter Estimation

The density function of (X, Y) is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{(\alpha_2+\alpha_3)\alpha_1}{k^2}\left(1+\frac{y}{k}\right)^{-(\alpha_2+\alpha_3)-1}\left(1+\frac{x}{k}\right)^{-\alpha_1-1} & for\ 0 < x < y < \infty \\ \frac{(\alpha_1+\alpha_3)\alpha_2}{k^2}\left(1+\frac{x}{k}\right)^{-(\alpha_1+\alpha_3)-1}\left(1+\frac{y}{k}\right)^{-\alpha_2-1} & for\ 0 < y < x < \infty \\ \frac{\alpha_3}{k}\left(1+\frac{x}{k}\right)^{-\alpha_1-1} & for\ x = y \end{cases} \tag{9}$$

For the bivariate life time distribution, we use the univariate censoring scheme given by Hanagal (1992a,1992b) because the individuals do not enter at the same time the study and withdrawal or death of an individual or termination of the study will censor both life times of the components. Here the censoring time is independent of the life times of both components. This is the standard univariate right.

Suppose that there are n independent pairs of components, for example, paired kidneys, lungs, eyes, ears in an individual under study and i^{-th} pair of the components have life times (X_i, Y_i) and censoring time (T_i). The life times associated with i^{-th} pair of the components are given by

$$(X_i,Y_i)=\begin{cases}(X_i,Y_i) & if\ max\ (X_i,Y_i)<T_i\\ (X_i,T_{iy}) & if\ X_i<T_{iy}<Y_i\\ (T_{ix},Y_i) & if\ Y_i<T_{ix}<X_i\\ (T_{ixy},T_{ixy}) & if\ min\ (X_i,Y_i)>T_{ixy}\end{cases}\qquad for\ i\ =\ 1,2,\dots,n \tag{10}$$

where T_{ix}, T_{iy} and T_{ixy} represent the unobserved random variables *x<y*, *x>y*, *x=y* respectively.

The likelihood of the sample of size n after discarding factors which do not contain any of the parameters of interest is given as follows

$$L(\underline{\alpha},t_{ix},t_{iy},t_{ixy})=\prod_{j=i}^{6}\prod_{i=1}^{n_j}f_{X,Y}^{j}\,(x_i,y_i) \tag{11}$$

where

$$f^1{}_{X,Y}(x,y)=\frac{(\alpha_2+\alpha_3)\alpha_1}{k^2}\left(1+\frac{y}{k}\right)^{-(\alpha_2+\alpha_3)-1}\left(1+\frac{x}{k}\right)^{-\alpha_1-1}\ for\ 0<x<y<t$$

$$f^2{}_{X,Y}(x,y)=\frac{(\alpha_1+\alpha_3)\alpha_2}{k^2}\left(1+\frac{x}{k}\right)^{-(\alpha_1+\alpha_3)-1}\left(1+\frac{y}{k}\right)^{-\alpha_2-1}\ for\ \ 0<y<x<t$$

$$f^3{}_{X,Y}(x,y)=\frac{\alpha_3}{k}\left(1+\frac{x}{k}\right)^{-\alpha_1-1}\qquad for\quad 0<\ x=y<t \tag{12}$$

$$f^4{}_{X,Y}(x,y)=\frac{(\alpha_2+\alpha_3)\alpha_1}{k^2}\left(1+\frac{t_y}{k}\right)^{-(\alpha_2+\alpha_3)-1}\left(1+\frac{x}{k}\right)^{-\alpha_1-1}\qquad for\ 0<x<t<t_y=y$$

$$f^5{}_{X,Y}(x,y)=\frac{(\alpha_1+\alpha_3)\alpha_2}{k^2}\left(1+\frac{t_x}{k}\right)^{-(\alpha_1+\alpha_3)-1}\left(1+\frac{y}{k}\right)^{-\alpha_2-1}\qquad for\ 0<y<t<t_x=x$$

$$f^6{}_{X,Y}(x,y)=\frac{\alpha}{k}\left(1+\frac{t_{xy}}{k}\right)^{-\alpha-1}\qquad for\quad 0<t<t_{xy}=min(x,y)$$

where $\sum_{j=1}^{6}n_j=n$, n_1,n_2,n_3,n_4,n_5 *and* n_6 be the numbers of realizations falling in the range corresponding to $f^1{}_{X,Y}(x,y), f^2{}_{X,Y}(x,y),\ f^3{}_{X,Y}(x,y)\ f^4{}_{X,Y}(x,y)$, $f^5{}_{X,Y}(x,y)$ and $f^6{}_{X,Y}(x,y)$ respectively. $f^1{}_{X,Y}(x,y)$, $f^2{}_{X,Y}(x,y)$, $f^4{}_{X,Y}(x,y)$ and $f^5{}_{X,Y}(x,y)$ are density functions with respect to the Lebesque measure on R^2, while , $f^3{}_{X,Y}(x,y)$, and $f^6{}_{X,Y}(x,y)$ are density functions with respect to the Lebesque measure on *R*.

Let the range of variability corresponding to $f^{j}{}_{X,Y}(x,y)$ be denoted by A_j, j=1, 2, …., 6.and $A_1, \dots, A_6$ are disjoint sets and letting $D_1 = \cup_{j=1}^{6} A_j$, $D_2 = A_2 \cup A_3$ and $D_3 = A_1 \cup A_2 \cup A_5$ then log-likelihood function $lnL_c(\underline{\alpha}, t_{ix}, t_{iy}, t_{ixy})$ can be written as follows:

$$\begin{aligned} ln\,L = (n_1+n_4)\,(ln(\alpha_2+\alpha_3)+ln\alpha_1)+(n_2+n_5)(ln(\alpha_1+\alpha_3)+ln\alpha_2)+n_3ln\alpha_3+nln\alpha \\ -(\alpha_1+1)\sum_{i\in D_1} l\,n\left(1+\frac{x_i}{k}\right)-\alpha_3\sum_{i\in D_2} l\,n\left(1+\frac{x_i}{k}\right)-\alpha_2\sum_{i\in A_3} l\,n\left(1+\frac{x_i}{k}\right) \\ -(\alpha_2+1)\sum_{i\in D_3} l\,n\left(1+\frac{y_i}{k}\right)-\alpha_3\sum_{i\in A_1} l\,n\left(1+\frac{y_i}{k}\right) \\ -(\alpha_1+\alpha_3+1)\sum_{i\in A_5} l\,n\left(1+\frac{t_{ix}}{k}\right)-(\alpha_2+\alpha_3+1)\sum_{i\in A_4} l\,n\left(1+\frac{t_{iy}}{k}\right) \\ -(\alpha+1)\sum_{i\in A_6} l\,n\left(1+\frac{t_{ixy}}{k}\right) \qquad (13) \end{aligned}$$

Thus, the following E-step and M- step are obtained

E- step:

The unobserved random variables, T_{ix}, T_{iy} and T_{ixy} follow the distributions as stated in (7). The conditional distributions of $T_x|\{(t_x,t,y)|(t_x>t>y)\}$, $T_y|\{(t_y,t,x)|(t_y>t>x)\}$ and $T_{xy}|\{(t_{xy},t)|(t_{xy}>t)\}$ are given as follows:

$$\begin{aligned} f_{T_x|(t_x,t,y)}(t_x) &= \frac{(\alpha_1+\alpha_3)}{k}\left(1+\frac{t_x}{k}\right)^{-(\alpha_1+\alpha_3)-1}\left(1+\frac{t}{k}\right)^{\alpha_1+\alpha_3} \quad ,y<t<t_x \\ f_{T_y|(t_y,t,x)}(t_y) &= \frac{(\alpha_2+\alpha_3)}{k}\left(1+\frac{t_y}{k}\right)^{-(\alpha_2+\alpha_3)-1}\left(1+\frac{t}{k}\right)^{\alpha_2+\alpha_3} \quad ,x<t<t_y \quad (14) \\ f_{T_{xy}|(t_{xy},t)}(t_{xy}) &= \frac{\alpha}{k}\left(1+\frac{t_{xy}}{k}\right)^{-\alpha-1}\left(1+\frac{t}{k}\right)^{\alpha} \quad ,t>t_{xy} \end{aligned}$$

The values of the first moments of the conditional unobserved random variables $\left(1+\frac{T_x}{k}\right)|\{(t_x,t,y)|(t_x>t>y)\}$, $\left(1+\frac{T_y}{k}\right)|\{(t_y,t,x)|(t_y>t>x)\}and\ (1+T_{xy}/k)|\{(t_{xy},t)|(t_{xy}>t)\}$ are as follows.

$$\begin{aligned} E(1+T_x/k)|\{(t_x,t,y)|(t_x>t>y)\} &= \frac{\alpha_1+\alpha_3}{\alpha_1+\alpha_3-1}\left(1+\frac{t}{k}\right)=b_1\left(1+\frac{t}{k}\right) \quad (say) \\ E(1+T_y/k)|\{(t_y,t,x)|(t_y>t>x)\} &= \frac{\alpha_2+\alpha_3}{\alpha_2+\alpha_3-1}\left(1+\frac{t}{k}\right)=b_2\left(1+\frac{t}{k}\right) \quad (say) \qquad (15) \\ E(1+T_{xy}/k)|\{(t_{xy},t)|(t_{xy}>t)\} &= \frac{\alpha}{\alpha-1}\left(1+\frac{t}{k}\right)=b_3\left(1+\frac{t}{k}\right) \quad (say) \end{aligned}$$

The conditional expectation $Q(\underline{\alpha}|\underline{\alpha}^{(k)})$ is obtained as follows:

$$\begin{aligned} Q(\underline{\alpha}|\underline{\alpha}^{(k)}) = (n_1+n_4)(ln(\alpha_2+\alpha_3)+ln\alpha_1)+(n_2+n_5)(ln(\alpha_1+\alpha_3)+ln\alpha_2)+n_3ln\alpha_3 \\ +n_6ln\alpha-(\alpha_1+1)\sum_{i\in D_1}x^*{}_i-\alpha_3\sum_{i\in D_2}x^*{}_i-\alpha_2\sum_{i\in A_3}x^*{}_i-(\alpha_2+1)\sum_{i\in D_3}y^*{}_i \\ -\alpha_3\sum_{i\in A_1}y^*{}_i-(\alpha_1+\alpha_3+1)\sum_{i\in A_5}\left(t^*{}_i+b_1{}^{(k)}\right) \\ -(\alpha_2+\alpha_3+1)\sum_{i\in A_4} l\,n\left(t^*{}_i+b_2{}^{(k)}\right) \\ -(\alpha+1)\sum_{i\in A_6}\left(t^*{}_i+b_3{}^{(k)}\right) \qquad (16) \end{aligned}$$

where $b_1{}^{(k)}, b_2{}^{(k)}, b_3{}^{(k)}$ are expressed in terms of $\alpha_1{}^{(k)}, \alpha_2{}^{(k)}, \alpha_3{}^{(k)}$ at k^{th} iteration, and

$$x^*_i = ln\left(1+\frac{x_i}{k}\right), \qquad y^*_i = ln\left(1+\frac{y_i}{k}\right) \ and \ \ t^*_i = ln\left(1+\frac{t_i}{k}\right).$$

M-step:

The following likelihood equations are obtained by equating the partial derivatives of $Q(\underline{\alpha}|\underline{\alpha}^{(k)})$ with respect to $\alpha_1, \alpha_2 and\ \alpha_3$ to zero:

$$\frac{n_1+n_4}{\alpha_1}+\frac{n_2+n_5}{\alpha_1+\alpha_3}+\frac{n_6}{\alpha}-\sum_{i\in D_1} x^*_i-\sum_{i\in A_5}\left(t^*_i+b_1{}^{(k)}\right)-\sum_{i\in A_6}\left(t^*_i+b_3{}^{(k)}\right)=0 \tag{17}$$

$$\frac{n_2+n_5}{\alpha_2}+\frac{n_1+n_4}{\alpha_2+\alpha_3}+\frac{n_6}{\alpha}-\sum_{i\in A_3} x^*_i-\sum_{i\in D_3} y^*_i-\sum_{i\in A_4}\left(t^*_i+b_2{}^{(k)}\right)-\sum_{i\in A_6}\left(t^*_i+b_3{}^{(k)}\right)=0 \tag{18}$$

$$\frac{n_2+n_5}{\alpha_1+\alpha_3}+\frac{n_1+n_4}{\alpha_2+\alpha_3}+\frac{n_3}{\alpha_3}+\frac{n_6}{\alpha}-\sum_{i\in A_3} x^*_i-\sum_{i\in A_1} y^*_i-\sum_{i\in A_5}\left(t^*_i+b_1{}^{(k)}\right)-\sum_{i\in A_4}\left(t^*_i+b_2{}^{(k)}\right)$$
$$-\sum_{i\in A_6}\left(t^*_i+b_3{}^{(k)}\right)=0 \tag{19}$$

The above likelihood equations are solved for the maximum likelihood estimates $(\hat{\alpha}_1{}^{(k)}, \hat{\alpha}_2{}^{(k)}, \hat{\alpha}_3{}^{(k)})$ using the Newton-Raphson procedure. Below the observed elements of the symmetric information matrix at the current guess $\underline{\alpha} = \underline{\alpha}^{(k)}$ are given which is required in the Newton-Raphson iterative procedure.

$$\begin{aligned}
\frac{\partial^2 \ln L}{\partial\alpha_1^2} &= -\frac{(n_1+n_4)}{\alpha_1{}^2}-\frac{(n_2+n_5)}{(\alpha_1+\alpha_3)^2}-\frac{n_6}{\alpha^2}\\
\frac{\partial^2 \ln L}{\partial\alpha_1\partial\alpha_2} &= -\frac{n_6}{\alpha^2}\\
\frac{\partial^2 \ln L}{\partial\alpha_1\partial\alpha_3} &= -\frac{(n_2+n_5)}{(\alpha_1+\alpha_3)^2}-\frac{n_6}{\alpha^2}\\
\frac{\partial^2 \ln L}{\partial\alpha_2^2} &= -\frac{(n_2+n_5)}{\alpha_2^2}-\frac{(n_1+n_4)}{(\alpha_2+\alpha_3)^2}-\frac{n_6}{\alpha^2}\\
\frac{\partial^2 \ln L}{\partial\alpha_2\partial\alpha_3} &= -\frac{(n_1+n_4)}{(\alpha_2+\alpha_3)^2}-\frac{n_6}{\alpha^2}\\
\frac{\partial^2 \ln L}{\partial\alpha_3^2} &= -\frac{(n_2+n_5)}{(\alpha_1+\alpha_3)^2}-\frac{(n_1+n_4)}{(\alpha_2+\alpha_3)^2}-\frac{n_3}{\alpha_3^2}-\frac{n_6}{\alpha^2}
\end{aligned} \tag{20}$$

As mentioned above the iterative process is carried out until the following condition is met by two subsequent solutions:

$$L\left(\underline{\alpha}^{(k+1)}, E\left[t_{ix}, t_{iy}, t_{ixy}|x_i, y_i, t_i\right]\right)-L\left(\underline{\alpha}^{(k)}, E\left[t_{ix}, t_{iy}, t_{ixy}|x_i, y_i, t_i\right]\right)<0.000001 \tag{21}$$

4. Simulation Study and Conclusions

The sample data are generated based on following algorithms:

- Step 1: Generate u_i using the Lomax distribution white parameters α_1, α_2, α_3.
- Step 2: Take $X = min(u_1, u_3)$ and $Y = min(u_2, u_3)$ and, therefore (X, Y) follows a bivariate exponential distribution of Marshall-Olkin type.
- Step 3: Generate t_i using the exponential distribution with failure rate θ, where $t_i s$ are the censoring times.

For two cases with respect to the $\alpha_i s$, we generated 1000 sets of samples. Each set consisted of three samples with sizes n = 20, 35 and 50. The corresponding maximum likelihood estimates are displayed in Table 1 together with the empirical standard deviation. The estimates denoted by MLE_{em} and SE_{em} are obtained by using the EM algorithm.

Parameters	α_1 1.8	α_2 1.7	α_3 1.5	α_1 0.8	α_2 0.6	α_3 0.9
	$n = 20$					
MLE_{em} SE_{em}	1.785 0.087	1.25 0.03	1.603 0.092	0.723 0.097	0.461 0.065	0.896 0.075
	$n = 35$					
MLE_{em} SE_{em}	1.734 0.067	1.455 0.025	1.1.503 0.073	0.69 0.059	0.408 0.025	0.872 0.052
	$n = 50$					
MLE_{em} SE_{em}	1.756 0.048	1.32 0.017	1.493 0.056	0.652 0.039	0.381 0.011	0.834 0.041

The estimates MLE_{em} are close to the true parameter values and the standard errors (SE_{em}) decrease as the sample size increases. The estimates for both methods are obtained by taking the mean of the 1000 maximum likelihood estimates and the mean of the 1000 standard deviations from the 1000 samples of size n = 20, 35, and 50. The estimates of the standard deviation of the maximum likelihood estimates of (α_1, α_2, α_3) are obtained by taking square root of the diagonal elements of the inverse of the observed Fisher information matrix.

One can also extend the estimation procedure based on the EM algorithm for other types of bivariate Lomax distribution like those of Freund (1961), Block-Basu(1974), or Proschan-Sullo(1974).

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