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Exponentiated Generalized Inverse Weibull Distribution

I. Elbatal and Hiba Z. Muhammed

Institute of Statistical Studies and Research,
Department of Mathematical Statistics, Cairo University, Egypt

Abstract

The inverse Weibull distribution can be readily applied to a wide range of situations including applications in medicine, reliability and ecology. In this article we introduce a new model of generalized inverse Weibull distribution referred to as the Exponentiated generalized inverse distribution. We provide a comprehensive mathematical treatment of this distribution. We derive the moment generating function and the rth moment thus generalizing some results in the literature. Expressions for the density, moment generating function and rth moment of the order statistics also are obtained. We discuss estimation of the parameters by maximum likelihood and provide the information matrix

Keywords: Inverse Weibull distribution; Hazard function; Order Statistics; Moments; Maximum likelihood estimation.

1 Introduction and Motivation

Statistical distributions are very useful in describing and predicting real world phenomena. Although many distributions have been developed, there are always rooms for developing distributions which are either more flexible or for fitting specific real world scenarios. This has motivated researchers seeking and developing new and more flexible distributions. As a result, many new distributions have been developed and studied. Mudholkar and Srivastava (1993) proposed the exponentiated Weibull distribution to analyze bathtub failure data. Gupta et al. (1998) first proposed a generalization of the standard exponential distribution, called the exponentiated exponential (EE) distribution, defined by the cumulative distribution function (cdf) $F(x) = (1-e^{-\lambda x})^{\alpha}$ for $x > 0$, $\alpha, \lambda > 0$. This equation is simply the power of the standard exponential cumulative distribution. For a full discussion and some of its mathematical properties, see Gupta and Kundu (2001). In a similar manner, Nadarajah and Kotz (2006) proposed the exponentiated gamma, exponentiated Fré chet and exponentiated Gumbel distributions, although the way they defined the cdf of the last two distributions is slightly different.

Cordeiro et al. (2013) proposed a new class of distributions that extend the exponentiated type distributions and they obtained some of its structural properties. Given a continuous cdf $G(x)$, they defined the exponentiated generalized (EG) class of distributions by

$$F(x) = \left[1-\{1-G(x)\}^{\alpha}\right]^{\beta}, \quad (1.1)$$

where $\alpha > 0$ and $\beta > 0$ are two additional shape parameters. We note that there is no complicated function in (1.1) in contrast with the beta generalized family (Eugene et al., 2002), which also includes two extra parameters but involves the beta incomplete function. (1.1) has tractable properties especially for simulations, since its quantile function takes a simple form, namely

$$x = Q_G\left[1-\left(1-u^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}}\right]$$

where $Q_G(u)$ is the baseline quantile function.

The baseline distribution $G(x)$ is clearly a special case of (1.1) when $\alpha = \beta = 1$. Setting $\alpha = 1$ gives the exponentiated type distributions de ned by Gupta et al. (1998). Further, the exponentiated exponential and exponentiated gamma distributions are obtained by taking $G(x)$ to be the exponential and gamma cumulative distributions, respectively. For $\beta = 1$ and if $G(x)$ is the Gumbel and Fréchet cumulative distributions, we obtain the exponentiated Gumbel and exponentiated Fréchet distributions, respectively, as defined by Nadarajah and Kotz (2006). Thus, the class of distributions (1.1) extends both exponentiated type distributions. The probability density function (pdf) of the new class has the form

$$f(x,\alpha,\beta) = \alpha\beta g(x)\left[1-G(x)\right]^{\alpha-1}\left[1-\{1-G(x)\}^{\alpha}\right]^{\beta-1}. \quad (1.2)$$

The exponentiated generalized (EG) family of densities (1.2) allows for greater exibility of its tails and can be widely applied in many areas of engineering and biology. Note that even if $g(x)$ is a symmetric distribution, the distribution $f(x)$ will not be a symmetric distribution. The two extra parameters in (1.2) can control both tail weights and possibly adding entropy to the center of the EG density function.

Hereafter, they defined the exponentiated-G ($Exp - G$ for short) distribution for an arbitrary parent distribution $G(x)$, say $X \sim EXP^c G$, if X has cumulative and density functions given by $H_c(x) = G(x)^c$ and $h_c(x) = c\ g(x)G(x)^{c-1}$, respectively. This is also called the Lehmann type I distribution, say, $Exp^c G$. For $c > 1$ and $c < 1$ and for larger values of x, the multiplicative factor $c\ G(x)^{c-1}$ is greater and smaller than one, respectively. The reverse assertion is also true for smaller values of x . The latter immediately implies that the ordinary moments associated with the density function $h_c(x)$ are strictly larger (smaller) than those associated with the density $g(x)$ when $c > 1$ $(c < 1)$. Note that there is a dual transformation $Exp^c(1-G)$ referred to as the Lehmann type II distribution corresponding to the cdf $F(x) = [1 - G(x)]^c$. Thus (1.1) encompasses both Lehmann type I ($Exp^\beta G$, for $\alpha = 1$) and Lehmann type II (($Exp^\alpha(1-G)$ for $\beta = 1$)) distributions (Lehmann, 1953). Clearly, the double construction $Exp^\beta\left[Exp^\alpha(1-G)\right]$ generates the EG class of distributions. The derivations of several properties of the EG class can be facilitated by this double transformation. The class of EG distributions shares an attractive physical interpretation whenever and are positive integers. Consider a device made of independent components in a parallel system. Furthermore, each component is made of independent subcomponents identically distributed according to $G(x)$ in a series system. The device fails if all components fail and each component fail if any subcomponent fails. Let $X_{j1},...,X_{j\alpha}$ denote the lifetimes of the subcomponents within the j_{th} component, $j = 1,...,\beta,$ with common cdf $G(x)$. Let X_j denote the lifetime of the j_{th} component and let X denote the lifetime of the device. Thus, the cdf of X is

$$P(X \le x) = P(X_1 \le x,..., X_\beta \le x) = P(X_1 \le x)^\beta$$

$$= 1 - P(X_1 > x)^\beta = [1 - P(X_{11} > x,..., X_{1\alpha} > x]^\beta$$

$$= \left[1 - P(X_{11} > x)^\alpha\right]^\beta = \left[1 - [1 - P(X_{11} \le x)]^\alpha\right]^\beta.$$

So, the lifetime of the device obeys the EG family of distributions. In this paper, we propose a new distribution based on the EG family, called Exponentiated Generalized Inverse Weibull ($EGIW$) distribution.

The inverse Weibull distribution is another life time probability distribution which can be used in the reliability engineering discipline. It can be used to model a variety of failure characteristics such as infant mortality, useful life and wear- out periods. It can also be used to determine the cost effectiveness, maintenance periods of reliability centered maintenance activities and applications in medicine, reliability and ecology. Keller and Kanath (1982) introduced the use of the inverse Weibull distribution as a suitable model to describe the degeneration phenomena of mechanical components such as the dynamic components (pistons, crankshaft, etc.) of diesel engines. The inverse Weibull distribution also provides a good fit to several data such as the times to breakdown of an insulating fluid, subject to the action of constant tension; see Nelson (1982). Calabria and Pulcini (1990) discussed the maximum likelihood and least squares estimation of its parameters. Calabria and Pulcini (1994) considered Bayes 2-sample prediction of the distribution. The random variable X has an inverse Weibull distribution if its cumulative distribution function (cdf) takes the form

$$G(x) = e^{-(\frac{\lambda}{x})^{\theta}}, x > 0, \theta, \lambda > 0, \quad (1.3)$$

Where λ is scale parameter and θ is shape parameter. The corresponding probability density function (pdf) is

$$g(x) = \theta\lambda^{\theta} x^{-\theta-1}\, e^{-(\frac{\lambda}{x})^{\theta}}, \quad (1.4)$$

The inverse Weibull distribution is also a limiting distribution of the largest order statistics. Drapella (1993) and Mudholkar and Kollia (1994) suggested the names complementary Weibull and reciprocal Weibull for the distribution (1.4).

The rest of the paper is organized as follows. In Section 2, we define the $EGIW$ distribution, discuss some special sub-models and provide its cumulative distribution function (cdf), the probability density function (pdf) and the hazard function. A formula for generating $EGIW$ random samples from the $EGIW$ distribution is given in Section 2. Section 3 discusses some important statistical properties of the $EGIW$ distribution a such as the quantile, the ordinary moments and measures of skewness and kurtosis. The distribution of the order statistics is expressed in Section 4 . Maximum likelihood estimates of the four parameter index to the distribution are presented in Section 5. Applications to three real data sets are performed in Section 6.

2. Exponentiated Generalized Inverse Weibull Distribution

In this section, we introduce the four- parameter Exponentiated Generalized Inverse Weibull $(EGIW)$ distribution. Using (1.3) in (1.1), the cdf of the $(EGIW)$ distribution can be written as

$$F(x) = \left[1 - \left\{1 - e^{-(\frac{\lambda}{x})^{\theta}}\right\}^{\alpha}\right]^{\beta} \quad (2.1)$$

The pdf of the new distribution can be written as

$$f(x) = \alpha\beta g(x)\left[1 - G(x)\right]^{\alpha-1}\left[1 - \left\{1 - G(x)\right\}^{\alpha}\right]^{\beta-1}$$

$$= \alpha\beta\theta\lambda^{\theta} x^{-\theta-1}\, e^{-(\frac{\lambda}{x})^{\theta}}\left[1 - e^{-(\frac{\lambda}{x})^{\theta}}\right]^{\alpha-1}\left[1 - \left\{1 - e^{-(\frac{\lambda}{x})^{\theta}}\right\}^{\alpha}\right]^{\beta-1}. \qquad (2.2)$$

Figures 1 and 2 illustrate plot the pdf and cdf of $(EGIW)$ distribution for selected values of the parameters.

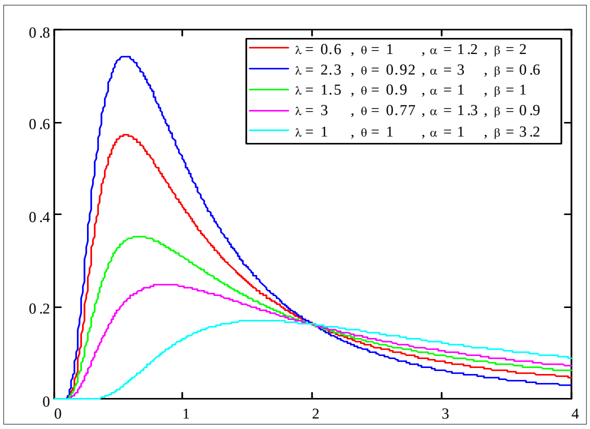

Figure 1: Plots of the probability density function of the (EGIW) distribution.

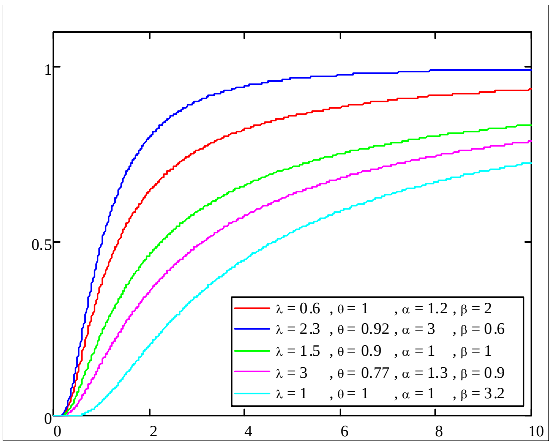

Figure 2: Plots of the cumulative distribution function of the (EGIW) distribution for some parameter values.

The survival and hazard (failure) rate functions of the $(EGIW)$ distribution are given by

$$\overline{F}(x) = 1 - \left[1 - \left\{1 - e^{-(\frac{\lambda}{x})^{\theta}}\right\}^{\alpha}\right]^{\beta}, \qquad (2.3)$$

and

$$h(x) = \frac{f(x)}{1 - F(x)}$$

$$= \frac{\alpha\beta\theta\lambda^{\theta}x^{-\theta-1}\, e^{-(\frac{\lambda}{x})^{\theta}}\left[1 - e^{-(\frac{\lambda}{x})^{\theta}}\right]^{\alpha-1}\left[1 - \left\{1 - e^{-(\frac{\lambda}{x})^{\theta}}\right\}^{\alpha}\right]^{\beta-1}}{1 - \left[1 - \left\{1 - e^{-(\frac{\lambda}{x})^{\theta}}\right\}^{\alpha}\right]^{\beta}}. \qquad (2.4)$$

respectively. Figures 3 and 4 illustrate plot survival and hazard (failure) rate functions of EGIW distribution for selected values of the parameters.

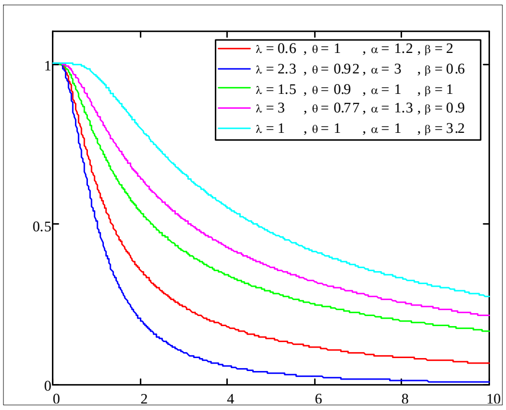

Figure 3: Plots of the survival function of the (EGIW) distribution.

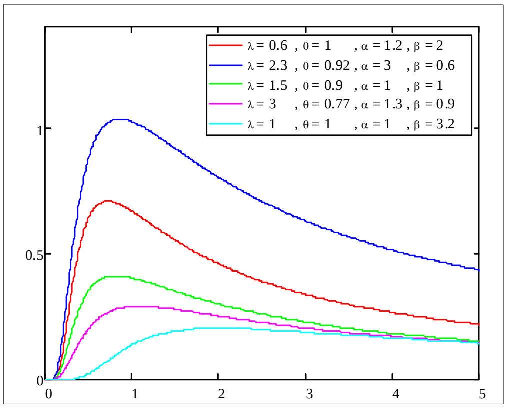

Figure 4: Plots of the hazard (failure) rate function of the (EGIW) distribution.

Special Cases of the EGIW Distribution

Exponentiated Generalized Inverse Weibull (EGIW) distribution is very flexible model that approaches to different distributions when its parameters are changed. The flexibility of the (EGIW) distribution is explained in the following. If X is a random variable with pdf (2.2), then we have the following cases**.**

2.1. Special Cases:

1- If $\alpha = \beta = 1$, then (2.2) reduces to the inverse Weibull distribution which is introduced by Gusmão et al. (2011).

2- If $\alpha = 1$ then we get the exponentiated (generalized) inverse Weibull distribution which is introduced by Gusmão et al. (2011).

3- If $\theta = 1$, then we get the exponentiated generalized inverse exponential.
4- If $\alpha = \beta = \theta = 1$, then we get the inverse exponential distribution.

2.2. Expansions for the Cumulative and Density Functions

In this subsection, we present some representations of cdf and pdf of $(EGIW)$. Equations (2.1) and (2.2) are straightforward to compute using any software with algebraic facilities. The mathematical relation given below will be useful in this subsection. If b is a positive real non integer and $|z| \le 1,$ we consider the power series expansion

$$(1-z)^{b-1} = \sum_{j=0}^{\infty} w_j z^j \qquad (2.5)$$

where

$$w_j = (-1)^j \binom{b-1}{j} = \frac{(-1)^j \Gamma b}{j!\Gamma(b-j)},$$

Applying (2.4) in (1.1) and using the binomial expansion for a positive real power yields

$$F(x) = \left[1 - \{1 - G(x)\}^{\alpha}\right]^{\beta}$$

$$= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{j+k} \binom{\beta}{j} \binom{\alpha j}{k} G(x)^k.$$

From Equation (2.1) we have

$$F(x) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{j+k} \binom{\beta-1}{j} \binom{\alpha j}{k} e^{-k(\frac{\lambda}{x})^{\theta}} \qquad (2.6)$$

And

$$f(x)=\alpha\beta\theta\lambda^{\theta}x^{-\theta-1}\,e^{-(\frac{\lambda}{x})^{\theta}}\left[1-e^{-(\frac{\lambda}{x})^{\theta}}\right]^{\alpha-1}\left[1-\left\{1-e^{-(\frac{\lambda}{x})^{\theta}}\right\}^{\alpha}\right]^{\beta-1}.$$

$$=C_{jk}\alpha\beta\theta\lambda^{\theta}x^{-\theta-1}e^{-(k+1)(\frac{\lambda}{x})^{\theta}}. \qquad (2.7)$$

where

$$C_{jk}=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\binom{\beta-1}{j}\binom{\alpha(j+1)-1}{k}.$$

3. Statistical Properties

This section is devoted to studying statistical properties of the $(EGIW)$ distribution, specifically quantile function, moments and moment generating function.

3.1. Quantile Function and Median

The quantile function corresponding to (2.1) is

$$Q(u)=F^{-1}(u)=\lambda\left[-\log\left[1-\left\{1-u^{\frac{1}{\beta}}\right\}^{\frac{1}{\alpha}}\right]\right]^{-\frac{1}{\theta}} \qquad (3.1)$$

Simulating the $(EGIW)$ random variable is straight forward. Let U be a uniform variate on the unit interval $(0,1)$ .Thus, by means of the inverse transformation method, we consider the random variable X given by

$$X=\lambda\left[-\log\left[1-\left\{1-u^{\frac{1}{\beta}}\right\}^{\frac{1}{\alpha}}\right]\right]^{-\frac{1}{\theta}}, \quad (3.2)$$

3.2. Moments

In this subsection we discuss the r_{th} moment for $(EGIW)$ distribution. Moments are necessary and important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution (e.g., tendency, dispersion, skewness and kurtosis).

Theorem (3.1).

If X has $EGIW(\phi,x)$, $\phi=(\lambda,\theta,\alpha,\beta)$ then the r_{th} moment of X is given by the following

$$\mu_r^{/}(x)=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\binom{\beta-1}{j}\binom{\alpha(j+1)-1}{k}\alpha\beta\lambda^{r}(k+1)^{\frac{r}{\theta}-1}\Gamma(1-\frac{r}{\theta}). \qquad (3.3)$$

Proof:

Let X be a random variable with density function (2.7). The r_{th} ordinary moment of the

$(EGIW)$ distribution is given by

$$\mu_r^{/}(x) = E(X^{r)} = \int_0^\infty x^r f(x,\phi)dx$$

$$= C_{jk}\alpha\beta\theta\lambda^\theta \int_0^\infty x^{r-\theta-1} e^{-(k+1)(\frac{\lambda}{x})^\theta} dx$$

$$= C_{jk}\alpha\beta\lambda^r (k+1)^{\frac{r}{\theta}-1}\Gamma(1-\frac{r}{\theta}) \qquad (3.4)$$

where

$$C_{jk} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\binom{\beta-1}{j}\binom{\alpha(j+1)-1}{k}$$

which completes the proof .

Based on the first four moments of the $(EGIW)$ distribution, we present the shortcomings of the classical kurtosis measure are well-known. There are many heavy tailed distributions for which this measure is infinite. So, it becomes uninformative precisely when it needs to be. The Bowley skewness (Kenney and Keeping, 1962) is based on quartiles

$$S_K = \frac{Q_{0.75} - 2Q_{0.5} + Q_{0.25}}{Q_{0.75} - Q_{0.25}}, \qquad (3.5)$$

And the Moors' kurtosis (1998) is based on octiles

$$K_u = \frac{Q_{0.875} - Q_{0.625} - Q_{0.375} + Q_{0.125}}{Q_{0.75} - Q_{0.25}}. \qquad (3.6)$$

Where $Q(.)$ represents the quantile function.

3.3. Moment generating function

In this subsection, we derived the moment generating function of $(EGIW)$ distribution.
Theorem (3.2): If X has $(EGIW)$ distribution, then the moment generating function $M_X(t)$ has the following form

$$M_X(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}C_{jk}\alpha\beta\lambda^r(k+1)^{\frac{r}{\theta}-1}\Gamma(1-\frac{r}{\theta}). \quad (3.7)$$

Proof.

We start with the well-known definition of the moment generating function given by $M_x(t) = E(e^{tX}) = \int_0^\infty e^{tx} f(x,\phi)dx$, since $\sum_{r=0}^{\infty}\frac{t^r}{r!}x^r f(x)$ converges and each term is integrable for all t close to 0, then we can rewrite the moment generating function as $M_x(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}E(X^r)$ by replacing $E(X^r)$ by the right side of Equation (3.3) the desired result is obtained.

4. Order Statistics

In this section, we derive closed form expressions for the pdfs of the i_{th} order statistic of the $(EGIW)$ distribution; also, the measures of skewness and kurtosis of the distribution of the i_{th} order statistic in a sample of size n for different choices of $n;i$ are presented in this section. Let $X_1, X_2,..., X_n$ be a simple random sample from $(EGIW)$ distribution with cdf and pdf given by (2.1) and (2.2), respectively. Let $X_{(1\,:\,n)} \le X_{(2\,:\,n)} \le .... \le X_{(n\,:\,n)}$ denote the order statistics obtained from this sample. We now give the probability density function of $X_{i\,:\,n}$, say $f_{i\,:\,n}(x)$ and the moments of $X_{i\,:\,n}, i=1,2,...,n$. Therefore, the measures of skewness and kurtosis of the distribution of the $X_{r\,:\,n}$ are presented. The probability density function of $X_{i\,:\,n}$ is given by

$$f_{i\,:\,n}(x)=\frac{1}{B(i,n-i+1)}\left[F(x,\phi)\right]^{i-1}\left[1-F(x,\phi)\right]^{n-i} f(x,\phi) \quad (4.1)$$

where $F(x,\phi)$ and $f(x,\phi)$ are the cdf and pdf of the $EGIW$ distribution given by (2.1) and (2.2) respectively, and $B(.,.)$ is the beta function. We have

$$f_{i\,:\,n}(x)=\frac{\alpha\beta}{B(i,n-i+1)}g(x)\{1-G(x)\}^{\alpha-1}\left[1-\{1-G(x)\}^{\alpha}\right]^{\beta i-1}$$

$$\times\left\{1-\left[1-\{1-G(x)\}^{\alpha}\right]^{\beta}\right\}^{n-i}, \qquad (4.2)$$

using the binomial series expansion, $f_{i\,:\,n}(x)$ can be expressed as

$$f_{i\,:\,n}(x)=\sum_{k=0}^{n-i}(-1)^k\binom{n-i}{k}\frac{\alpha\beta}{B(i,n-i+1)}g(x)\{1-G(x)\}^{\alpha-1}$$

$$\times\left[1-\{1-G(x)\}^{\alpha}\right]^{\beta(i+k)-1}, \quad (4.3)$$

again applying (2.5) to the last term , we get

$$f_{i\,:\,n}(x)=\sum_{k=0}^{n-i}\sum_{r=0}^{\infty}(-1)^{k+r}\binom{n-i}{k}\binom{\beta(i+k)-1}{r}\frac{\alpha\beta}{B(i,n-i+1)}g(x)\{1-G(x)\}^{\alpha(r+1)-1}$$

$$=\sum_{k=0}^{n-i}\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{k+r+m}\binom{n-i}{k}\binom{\beta(i+k)-1}{r}\binom{\alpha(r+1)-1}{m}$$

$$\times\frac{\alpha\beta}{B(i,n-i+1)}g(x)G(x)^{m}$$

$$=w_{krm}\frac{\alpha\beta}{B(i,n-i+1)}g(x)G(x)^{m}. \qquad (4.4)$$

where

$$w_{krm}=\sum_{k=0}^{n-i}\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{k+r+m}\binom{n-i}{k}\binom{\beta(i+k)-1}{r}\binom{\alpha(r+1)-1}{m}.$$

substituting from (2.1) and (2.2) into (4.4), we can express the j_{th} ordinary moment of the i_{th} order statistics $X_{i\,:\,n}$ say $E(X_{i\,:\,n}^{j})$ as a liner combination of the j_{th} moments of the $(EGIW)$ distribution with different shape parameters.

$$\mu_{i\,:\,n}^{(j)}=w_{krm}\frac{1}{B(i,n-i+1)}\alpha\beta\theta\lambda^{\theta}\int_{0}^{\infty}x^{j-\theta-1}\,e^{-(m+1)(\frac{\lambda}{x})^{\theta}}dx$$

$$=w_{krm}\frac{\alpha\beta\lambda^{j}}{B(i,n-i+1)(m+1)^{1-\frac{j}{\theta}}}\Gamma(1-\frac{j}{\theta}). \quad (4.5)$$

5. Estimation and Inference

In this section, we determine the maximum likelihood estimates (MLEs) of the parameters of the $EGIW$ distribution from complete samples only. Let $X_1, X_2, ..., X_n$ be a random sample of size n from $EGIW\ (\phi, x)$, $\phi=(\lambda,\theta,\alpha,\beta)$. The log likelihood function for the vector of parameters $\phi=(\lambda,\theta,\alpha,\beta)$ can be written as

$$\log L = n\log\alpha + n\log\beta + n\log\theta + +n\theta\log\lambda$$

$$-(\theta+1)\sum_{i=1}^{n}\log(x_i) - \sum_{i=1}^{n}(\frac{\lambda}{x_i})^{\theta} + (\alpha-1)\sum_{i=1}^{n}\log\left[1-e^{-(\frac{\lambda}{x_i})^{\theta}}\right]$$

$$+(\beta-1)\sum_{i=1}^{n}\log\left[1-\left\{1-e^{-(\frac{\lambda}{x_i})^{\theta}}\right\}^{\alpha}\right] \qquad (5.1)$$

The log-likelihood can be maximized either directly or by solving the nonlinear likelihood equations obtained by differentiating (5.1). The components of the score vector are given by

$$\frac{\partial\log L}{\partial\alpha} = \frac{n}{\alpha}\sum_{i=1}^{n}\log\left[1-e^{-(\frac{\lambda}{x_i})^{\theta}}\right]$$

$$+(\beta-1)\sum_{i=1}^{n}\frac{\left\{1-e^{-(\frac{\lambda}{x_i})^{\theta}}\right\}^{\alpha}\log\left[1-e^{-(\frac{\lambda}{x_i})^{\theta}}\right]}{\left[1-\left\{1-e^{-(\frac{\lambda}{x_i})^{\theta}}\right\}^{\alpha}\right]}, \quad (5.2)$$

$$\frac{\partial\log L}{\partial\theta} = \frac{n}{\theta} - \sum_{i=1}^{n}\log(x_i) + \sum_{i=1}^{n}(\frac{\lambda}{x_i})^{\theta}\log(\frac{\lambda}{x_i})$$

$$+(\alpha-1)\sum_{i=1}^{n}\frac{\log(\frac{\lambda}{x_i})(\frac{\lambda}{x_i})^{\theta}e^{-(\frac{\lambda}{x_i})^{\theta}}}{\left[1-e^{-(\frac{\lambda}{x_i})^{\theta}}\right]}$$

$$+\alpha(\beta-1)\sum_{i=1}^{n}\frac{\log(\frac{\lambda}{x_i})(\frac{\lambda}{x_i})^{\theta}e^{-(\frac{\lambda}{x_i})^{\theta}}\left\{1-e^{-(\frac{\lambda}{x_i})^{\theta}}\right\}^{\alpha-1}}{\left[1-\left\{1-e^{-(\frac{\lambda}{x_i})^{\theta}}\right\}^{\alpha}\right]}, \qquad (5.3)$$

$$\frac{\partial\log L}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\log\left[1-\left\{1-e^{-(\frac{\lambda}{x_i})^{\theta}}\right\}^{\alpha}\right], \ (5.4)$$

and

$$\frac{\partial \log L}{\partial \lambda}=\frac{n\theta}{\lambda}-\theta\sum_{i=1}^{n}(\frac{1}{x_i})(\frac{\lambda}{x_i})^{\theta-1}+\theta(\alpha-1)\sum_{i=1}^{n}\frac{(\frac{1}{x_i})(\frac{\lambda}{x_i})^{\theta-1}e^{-(\frac{\lambda}{x_i})^{\theta}}}{\left[1-e^{-(\frac{\lambda}{x_i})^{\theta}}\right]}$$

$$+\alpha\theta(\beta-1)\sum_{i=1}^{n}\frac{(\frac{1}{x_i})(\frac{\lambda}{x_i})^{\theta-1}e^{-(\frac{\lambda}{x_i})^{\theta}}\left\{1-e^{-(\frac{\lambda}{x_i})^{\theta}}\right\}^{\alpha-1}}{\left[1-\left\{1-e^{-(\frac{\lambda}{x_i})^{\theta}}\right\}^{\alpha}\right]} \quad (5.5)$$

We can find the estimates of the unknown parameters by maximum likelihood method by setting these above non-linear equations (5.2) - (5.5) to zero and solve them simultaneously. Therefore, we have to use mathematical package to get the MLE of the unknown parameters. Therefore, we have to use mathematical package to get the MLE of the unknown parameters. Also, all the second order derivatives exist. Thus we have the inverse dispersion matrix is given by

$$\begin{pmatrix}\hat{\lambda}\\ \hat{\theta}\\ \hat{\alpha}\\ \hat{\beta}\end{pmatrix}\sim N\left[\begin{pmatrix}\lambda\\ \theta\\ \alpha\\ \beta\end{pmatrix},\begin{pmatrix}\widehat{V_{\lambda\lambda}} & \widehat{V_{\lambda\theta}} & \widehat{V_{\lambda\alpha}} & \widehat{V_{\lambda\beta}}\\ \widehat{V_{\theta\lambda}} & \widehat{V_{\theta\theta}} & \widehat{V_{\theta\alpha}} & \widehat{V_{\theta\beta}}\\ \widehat{V_{\alpha\lambda}} & \widehat{V_{\alpha\theta}} & \widehat{V_{\alpha\alpha}} & \widehat{V_{\alpha\beta}}\\ \widehat{V_{\beta\lambda}} & \widehat{V_{\beta\theta}} & \widehat{V_{\beta\alpha}} & \widehat{V_{\beta\beta}}\end{pmatrix}\right]$$

$$V^{-1}=-E\begin{pmatrix}V_{\lambda\lambda} & V_{\lambda\theta} & V_{\lambda\alpha} & V_{\lambda\beta}\\ V_{\theta\lambda} & V_{\theta\theta} & V_{\theta\alpha} & V_{\theta\beta}\\ V_{\alpha\lambda} & V_{\alpha\theta} & V_{\alpha\alpha} & V_{\alpha\beta}\\ V_{\beta\lambda} & V_{\beta\theta} & V_{\beta\alpha} & V_{\beta\beta}\end{pmatrix} \quad (5.6)$$

where

$$V_{\lambda\lambda}=\frac{\partial^2 L}{\partial\lambda^2},V_{\theta\theta}=\frac{\partial^2 L}{\partial\theta^2},V_{\alpha\alpha}=\frac{\partial^2 L}{\partial\alpha^2},V_{\beta\beta}=\frac{\partial^2 L}{\partial\beta^2}$$

$$V_{\lambda\alpha}=\frac{\partial^2 L}{\partial\alpha\partial\lambda},V_{\beta\theta}=\frac{\partial^2 L}{\partial\beta\partial\theta},V_{\alpha\theta}=\frac{\partial^2 L}{\partial\alpha\partial\theta},V_{\alpha\beta}=\frac{\partial^2 L}{\partial\alpha\partial\beta},V_{\beta\lambda}=\frac{\partial^2 L}{\partial\beta\partial\lambda}.$$

By solving this inverse dispersion matrix these solutions will yield asymptotic variance and covariances of these ML estimators for $\hat{\lambda},\hat{\theta},\hat{\alpha}$ and $\hat{\beta}$. Using (5.6), we approximate $100(1-\gamma)\%$ confidence intervals for λ,θ,α and β are determined respectively as

$$\hat{\lambda} \pm z_{\frac{\gamma}{2}} \sqrt{\widehat{V_{\lambda\lambda}}}\,,\ \ \hat{\theta} \pm z_{\frac{\gamma}{2}} \sqrt{\widehat{V_{\theta\theta}}}\,, \hat{\alpha} \pm z_{\frac{\gamma}{2}} \sqrt{\widehat{V_{\alpha\alpha}}} \text{ and } \hat{\beta} \pm z_{\frac{\gamma}{2}} \sqrt{\widehat{V_{\beta\beta}}}$$

where z_{γ} is the upper $100\gamma_{the}$ percentile of the standard normal distribution.

6. Application

In this section, the flexibility and potentiality of the EGIW distribution are examined using three real data sets. We provide an application of the EGIW distribution and their sub-models: generalized inverse Weibull and inverse Weibull distributions. The first data set given by Lee and Wang (2003) which represent remission times (in months) of a random sample of 128 bladder cancer patients. The data are as follows: 0.08, 2.09, 3.48, 4.87, 6.94 , 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46 , 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69. We estimate the unknown parameters of each distribution by the maximum-likelihood method, and the values for some models of the following statistics: Kolmogorov-Smirnov (K-S) statistic (the distance between the empirical CDFs and the fitted CDFs), Akaike information criterion (AIC), Bayesian information criterion (BIC), and the consistent Akaike information criterion (CAIC) are used to compare the candidate distributions. The best distribution corresponds to lower -2logL, AIC, BIC, CAIC statistics value.

Table 1. Maximum-likelihood estimates, AIC , BIC and CAIC values, and Kolmogorov- Smirnov statistics for the 128 bladder cancer patients data.

The Model	**MLEs**				**Measures**				
	α	θ	β	λ	K_S	-2logL	AIC	BIC	AICC
IW		0.464		16.142	0.503	1000.238	1004.238	1009.942	1004.334
GIW		0.34	1.797	0.75	0.369	990.362	996.362	1004.918	996.555
EGIW	2	0.5	1.05	1.006	0.608	976.092	984.092	995.5	984.417

From table 1, we observe that the EGIW distribution is a competitive distribution compared with other distributions. In fact, based on the values of the *AIC* and *BIC* criteria as well as the value of the *KS* -statistic, we observe that the EGIW distribution provides the best fit for these data among all the models considered.

The second data set have been obtained from Areset (1987) and it is provided below. It represents the lifetimes of 50 devices. 0.1, 0.2, 1, 1, 1, 1, 1, 2, 3, 6, 7, 1,1, 12, 18, 18, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55, 60, 63, 63, 67, 67, 67, 67, 72, 75, 79, 82, 82, 83, 84, 84, 84, 85, 85, 85, 85, 85, 86, 86.

Table2. Maximum-likelihood estimates, AIC, BIC and CAIC values, and Kolmogorov--Smirnov statistics for the lifetime of 50 devices.

The Model	MLEs				Measures				
	α	θ	β	λ	K_S	-2logL	AIC	BIC	CAIC
IW		0.397		1.043	0.435	562.14	566.14	569.964	566.396
GIW		0.274	1.273	0.596	0.324	574.951	580.951	586.687	581.473
EGIW	0.75	0.61	2.142	1.008	0.254	509.839	517.839	525.487	518.727

The third data set given by Abouammoh et al. (1994) which represent 40 patients suffering from leukemia from one of the Ministry of Health Hospitals in Saudi Arabia.

115 ,181, 255, 418, 441, 461, 516, 739, 743 ,789 ,807, 865, 924, 983, 1024,1062,

1063,1165, 1191,1222,1251,1277, 1290 ,1357,1369, 1408 ,1455, 1478, 1222

1549, 1578, 1578, 1599, 1603, 1605, 1696, 1735, 1799, 1815 ,1852.

Table 3. Maximum-likelihood estimates, AIC, BIC and CAIC values, and Kolmogorov--Smirnov statistics for the lifetime of 40 patients.

The Model	MLEs				Measures				
	α	θ	β	λ	K_S	-2logL	AIC	BIC	CAIC
IW		0.244		3.882	0.626	793.118	797.118	800.496	797.443
GIW		0.298	0.792	3.261	0.737	795.582	801.582	806.648	802.248
EGIW	0.933	0.453	1.09	1.102	0.831	709.471	717.471	724.226	718.614

Again, the values in Table 3 show parameter MLEs, the values of K_S, -2logL, AIC, BIC, AICC statistics for the three data set consecutively. From the above results, it is evident that the EGIW distribution is the best distribution for fitting these data sets compared to other distributions considered here, and it is a strong competitor to other distributions commonly used in literature for fitting lifetime data.

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