Applied Mathematical Sciences, Vol. 9, 2015, no. 23, 1139 - 1143
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ams.2015.411987

The Reduction of Order on Cauchy-Euler Equation with a Bulge Function

P. Haarsa[1] and S. Pothat[2]

[1]Department of Mathematics, Srinakharinwirot
Bangkok 10110, Thailand

[2]Wad Ban-Koh School, Bandara, Amphoe Pichai
Uttaradit 53220, Thailand

Abstract

In this paper, we study the nonhomogeneous second order differential equation of Cauchy-Euler equation with a bulge function. The reduction of order and Taylor series expansion of a bulge function are used to obtain the general solution.

Mathematics Subject Classification: 34M25, 34A12, 34K17

Keywords: Reduction of order, Cauchy-Euler equations, Bulge function

1 Introduction

There are more than one method to solve the Cauchy-Euler equation such as the Laplace transform and the method of reduction of order. H. Kim [2] applied the Laplace transform to find the solution of a homogeneous Cauchy-Euler equation of the second order ODE. The research is to make an application to its difference equation and oscillation. M. S. Abualrub [3] found a special case of a non-homogenous linear Euler-Cauchy ODE. In this paper, we study the nonhomogeneous second order differential equation of Cauchy-Euler equation with a bulge function by using the reduction of order.

2 Preliminaries

We begin our study by giving out the method of reduction of order, the Cauchy-Euler equation and the Taylor series expansion which can be used in this study. **The method of reduction of order** [1]. We consider the second order equation

$$y'' + P(x)y' + Q(x)y = 0. \tag{1}$$

on an open interval I on which P and Q are continuous. Suppose that we know one solution y_1 of equation (1). There exist a second linearly independent solution y_2 for which $y_2 = u(x)y_1$. We begin by substituting this expression in equation (1), using derivatives

$${y_2}' = u{y_1}' + u'y_1\,, \quad {y_2}'' = u{y_1}'' + 2u'{y_1}' + \ u''y_1. \tag{2}$$

We have

$$[u{y_1}'' + 2u'{y_1}' + \ u''y_1] + P\,[u{y_1}' + u'y_1\,] + Quy_1 = 0. \tag{3}$$

By rearranging equation (3), we derive

$$u\,[{y_1}'' + P{y_1}' + Qy_1] + u''y_1 + 2u'{y_1}' + Pu'y_1 = 0. \tag{4}$$

Since y_1 is a solution of equation (1), the bracketed expression in the above equation vanishes. It leaves the equation to

$$u''y_1 + (2{y_1}' + Py_1)\,u' = 0. \tag{5}$$

If we write $v = u'$, then $v' = u''$ and assume that y_1 never vanishes on the interval, then equation (5) becomes

$$v' + \left(2\frac{{y_1}'}{y_1} + P(x)\right)v = 0. \tag{6}$$

see further in [1]

The Cauchy-Euler equation. [1] An equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1}x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_{n-2}x^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + ... + a_1 x\frac{dy}{dx} + a_0 y = 0. \tag{7}$$

is called the Cauchy-Euler equation. It is also known as Euler equation and as the equidimensional equation. In this study, we study the nonhomogeneous second order differential equation of Cauchy-Euler equation with a bulge function in the form $\zeta^2 y'' + a\zeta y' + by = e^{-\frac{(\zeta-l)^2}{2}}$.

The Taylor series expansion of $e^{-\frac{(x-l)^2}{2}}$ is of the form

$$e^{-\frac{(x-l)^2}{2}} = e^{-\frac{l^2}{2}} + e^{-\frac{l^2}{2}}lx + e^{-\frac{l^2}{2}}\left(-\frac{1}{2} + \frac{l^2}{2}\right)x^2 + e^{-\frac{l^2}{2}}\left(-\frac{1}{2} + \frac{l^3}{6}\right)x^3. \tag{8}$$

3 Main Result

Lemma 3.1. *The general solution of a nonhomogeneous differential equation of Cauchy-Euler equation with a bulge function*

$$\zeta^2 y'' + a\zeta y' + by = e^{-\frac{(\zeta - l)^2}{2}} \tag{9}$$

where a, b are constants, l is a positive constant, and $y(\zeta)$ is unknown function can be expressed by

$$\begin{aligned} y(\zeta) &= \frac{1}{\zeta} + \frac{e^{-\frac{l^2}{2}}}{(a-2)} + \frac{e^{-\frac{l^2}{2}} l}{2(a-1)}\zeta + \frac{e^{-\frac{l^2}{2}}}{3a}\left(-\frac{1}{2} + \frac{l^2}{2}\right)\zeta^2 \\ &\quad + \frac{e^{-\frac{l^2}{2}}}{4(a+1)}\left(-\frac{1}{2} + \frac{l^3}{6}\right)\zeta^3 + c_1 + \frac{c_2}{\zeta}. \end{aligned}$$

Proof. Suppose that one of the solutions of the above equation is $y_1 = 1/\zeta$. By the reduction of order method, we suppose the second solution of the above solution is $y_2 = y_1 u(\zeta) = \frac{u}{\zeta}$. By taking derivative to this expression, we derive ${y_2}' = \frac{u'}{\zeta} - \frac{u}{\zeta^2}$ and ${y_2}'' = \frac{u''}{\zeta} - \frac{2u'}{\zeta^2} + \frac{2u}{\zeta^3}$. Substituting these two expressions to equation (9), it yields

$$\begin{aligned} \zeta^2\left[\frac{u''}{\zeta} - \frac{2u'}{\zeta^2} + \frac{2u}{\zeta^3}\right] + a\zeta\left[\frac{u'}{\zeta} - \frac{u}{\zeta^2}\right] + \frac{bu}{\zeta} &= e^{-\frac{l^2}{2}} + e^{-\frac{l^2}{2}} l\zeta \\ &\quad + e^{-\frac{l^2}{2}}\left(-\frac{1}{2} + \frac{l^2}{2}\right)\zeta^2 + e^{-\frac{l^2}{2}}\left(-\frac{1}{2} + \frac{l^3}{6}\right)\zeta^3. \end{aligned} \tag{10}$$

Let $\Psi = e^{-\frac{l^2}{2}} + e^{-\frac{l^2}{2}} l\zeta + e^{-\frac{l^2}{2}}\left(-\frac{1}{2} + \frac{l^2}{2}\right)\zeta^2 + e^{-\frac{l^2}{2}}\left(-\frac{1}{2} + \frac{l^3}{6}\right)\zeta^3$. Then, equation (10) becomes

$$\zeta u'' + (a-2)u' + \frac{(b-a+2)u}{\zeta} = \Psi. \tag{11}$$

with the condition $b - a + 2 = 0$, equation (11) yields

$$u'' + \frac{(a-2)}{\zeta}u' = \frac{\Psi}{\zeta}. \tag{12}$$

Let $v = u'$, then $v' = u''$. Substituting to equation (12), we have

$$v' + \frac{(a-2)}{\zeta}v = \frac{\Psi}{\zeta}. \tag{13}$$

which is the linear first-order equation such that $P(\zeta) = \frac{(a-2)}{\zeta}$ and $Q(\zeta) = \frac{\Psi}{\zeta}$ on an interval on which the coefficient functions $P(\zeta)$ and $Q(\zeta)$ are continuous.

We multiply each side in equation (13) by the integrating factor $IF = \zeta^{(a-2)}$. The result is

$$\zeta^{(a-2)}\left[v' + \frac{(a-2)}{\zeta}v\right] = \zeta^{(a-2)}\left[\frac{\Psi}{\zeta}\right]. \tag{14}$$

The left-hand side is the derivative of the product $\zeta^{(a-2)}.y(\zeta)$. Therefore, equation (14) is equivalent to $d\left(\zeta^{(a-2)}v\right) = \Psi\zeta^{(a-2)}$. Integrating both sides of this expression with respect to ζ, we derive

$$\begin{aligned}\zeta^{(a-2)}v &= \int\left[e^{-\frac{l^2}{2}} + e^{-\frac{l^2}{2}}l\zeta + e^{-\frac{l^2}{2}}\left(-\frac{1}{2}+\frac{l^2}{2}\right)\zeta^2\right.\\ &\quad \left.+e^{-\frac{l^2}{2}}\left(-\frac{1}{2}+\frac{l^3}{6}\right)\zeta^3\right]\zeta^{(a-3)}d\zeta. \end{aligned}\tag{15}$$

Therefore, from the above equation, we have

$$v = \frac{e^{-\frac{l^2}{2}}}{(a-2)} + \frac{e^{-\frac{l^2}{2}}l}{(a-1)}\zeta + \frac{e^{-\frac{l^2}{2}}}{a}\left(-\frac{1}{2}+\frac{l^2}{2}\right)\zeta^2 + \frac{e^{-\frac{l^2}{2}}}{(a+1)}\left(-\frac{1}{2}+\frac{l^3}{6}\right)\zeta^3 + c_1.$$

Since $u' = v$, and by integrating with respect to ζ, we have

$$\begin{aligned}u &= \frac{e^{-\frac{l^2}{2}}}{(a-2)}\zeta + \frac{e^{-\frac{l^2}{2}}l}{2(a-1)}\zeta^2 + \frac{e^{-\frac{l^2}{2}}}{3a}\left(-\frac{1}{2}+\frac{l^2}{2}\right)\zeta^3\\ &\quad +\frac{e^{-\frac{l^2}{2}}}{4(a+1)}\left(-\frac{1}{2}+\frac{l^3}{6}\right)\zeta^4 + c_1\zeta + c_2. \end{aligned}\tag{16}$$

Substituting u into $y_2 = y_1u(\zeta) = \frac{u}{\zeta}$, we derive

$$\begin{aligned}y_2 &= \frac{e^{-\frac{l^2}{2}}}{(a-2)} + \frac{e^{-\frac{l^2}{2}}l}{2(a-1)}\zeta + \frac{e^{-\frac{l^2}{2}}}{3a}\left(-\frac{1}{2}+\frac{l^2}{2}\right)\zeta^2\\ &\quad +\frac{e^{-\frac{l^2}{2}}}{4(a+1)}\left(-\frac{1}{2}+\frac{l^3}{6}\right)\zeta^3 + c_1 + \frac{c_2}{\zeta}. \end{aligned}\tag{17}$$

Since y_1 and y_2 are linearly independent, the general solution of equation (9) is

$$\begin{aligned}y(\zeta) &= \frac{1}{\zeta} + \frac{e^{-\frac{l^2}{2}}}{(a-2)} + \frac{e^{-\frac{l^2}{2}}l}{2(a-1)}\zeta + \frac{e^{-\frac{l^2}{2}}}{3a}\left(-\frac{1}{2}+\frac{l^2}{2}\right)\zeta^2\\ &\quad +\frac{e^{-\frac{l^2}{2}}}{4(a+1)}\left(-\frac{1}{2}+\frac{l^3}{6}\right)\zeta^3 + c_1 + \frac{c_2}{\zeta}. \end{aligned}\tag{18}$$

□

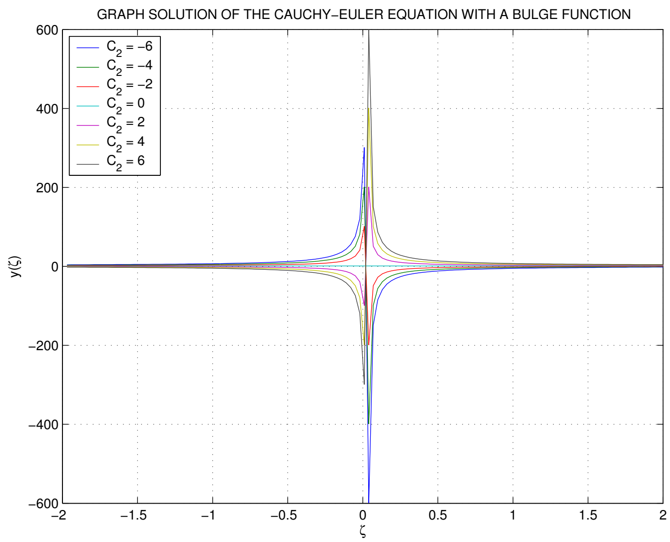

Figure 1: show several different solution curves with $c_1 = 1$, with different values of c_2, and by assigning values $l = 2$ and $a = 3$.

4 Conclusion

In this paper, we study the nonhomogeneous second order differential equation of Cauchy-Euler equation with a bulge function which is denoted by $f(\zeta) = e^{-\frac{(\zeta-l)^2}{2}}$ where l is a positive constant. H. Kim [2] found the solution of Euler-Cauchy equation expressed by differential operator of homogeneous second order differential equation using Laplace transform. For this study, we use the method of the reduction of order to solve the nonhomogeneous second order differential equation of Cauchy-Euler equation with a bulge function. We apply the Taylor series expansion to the bulge function to obtain the solution.

References

[1] C. Henry Edwards and David E. Penney, *Differential Equations and Boundary Value Problem*, Pearson Education, Inc, USA, 2004.

[2] H. Kim, The solution of Euler-Cauchy equation expressed by differential operator using Laplace transform, *IJPAM*,**84**(4)(2013), 345 - 351. http://dx.doi.org/10.12732/ijpam.v84i4.4

[3] M. S. Abualrub, A formula for solving a special case of Euler-Cauchy ODE, *International Math. Forum*, **4**(40)(2009), 1997 - 2000.

Received: December 8, 2014; Published: February 10, 2015