Applied Mathematical Sciences, Vol. 9, 2015, no. 93, 4635 - 4642
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ams.2015.56416

A Note on the Degenerate High Order Daehee Polynomials

Jin-Woo Park

Department of Mathematics Education
Daegu University, Gyeongsan, 712-714, Korea

Jongkyum Kwon[1]

Department of Mathematics
Kyungpook National University
Daegu, 702-701, Korea

Abstract

In this paper, we consider the degenerate high order Daehee polynomials which are derived from p-adic invariant integral on $\mathbb{Z}_p$ and investigate some properties of those polynomials.

Mathematics Subject Classification: 05A10, 05A19

Keywords: Bernoulli polynomial of order r, Daehee polynomials, degenerate high order Daehee polynomials

1. Introduction

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. The p-adic norm is normally defined by $|p|_p = \frac{1}{p}$. Let $f(x)$ be a uniformly differentiable function on $\mathbb{Z}_p$. Then the

[1]Corresponding author

p-adic invariant integral on $\mathbb{Z}_p$ is defined as

$$\begin{aligned}\int_{\mathbb{Z}_p} f(x)d\mu_0(x) &= \lim_{N\to\infty}\sum_{x=0}^{p^N-1} f(x)d\mu_0\left(x+p^N\mathbb{Z}_p\right)\\ &= \lim_{N\to\infty}\frac{1}{p^N}\sum_{n=0}^{p^N-1} f(x),\ \text{(see [12, 13, 15])}.\end{aligned} \tag{1.1}$$

From (1.1), we have

$$I_0(f_1)-I_0(f)=f^{'}(0). \tag{1.2}$$

As is well known, the *Bernoulli polynomials* are defined by the generating function to be

$$\left(\frac{t}{e^t-1}\right)^r e^{xt}=\sum_{n=0}^{\infty}B_n^{(r)}(x)\frac{t^n}{n!},\ \text{(see [1-2, 4-12])}. \tag{1.3}$$

When $x=0$, $B_n=B_n(0)$, $(n\geq 0)$, are called the *ordinary Bernoulli numbers.*

In [2], L. Carlitz consider the degenerate Bernoulli polynomials which are given by the generating function to be

$$\frac{t}{(1+\lambda t)^{\frac{x}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\beta_n(x|\lambda)\frac{t^n}{n!}. \tag{1.4}$$

When $x=0$, $\beta_n(\lambda)=\beta_n(0|\lambda)$ are called the *degenerate Bernoulli numbers.* Note that $\lim_{\lambda\to 0}\beta_n(\lambda)=B_n$.

The *Daehee polynomials of order* r are defined by

$$D_n(x)=\int_{\mathbb{Z}_p}(x+y)_n d\mu_0(y),\ (n\geq 0),\ \text{(see [6, 9, 10])}. \tag{1.5}$$

From (1.2) and (1.5), we can derive the generating function to be

$$\begin{aligned}\sum_{n=0}^{\infty}D_n^{(r)}(x)\frac{t^n}{n!} &=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{x_1+\cdots+x_r+x}d\mu_0(x_1)\cdots d\mu_0(x_r)\\ &=\left(\frac{\log(1+t)}{t}\right)^r(1+t)^x,\end{aligned} \tag{1.6}$$

(see [6, 9]).

By (1.3) and (1.6), it is not difficult to show that

$$\left(\frac{\log(1+t)}{t}\right)^r(1+t)^{x-1}=\sum_{n=0}^{\infty}B_n^{(n+r+1)}(x)\frac{t^n}{n!},\ \text{(see [6])}. \tag{1.7}$$

In this paper, we consider the degenerate high order Daehee numbers and polynomials which are derived from p-adic invariant integral on $\mathbb{Z}_p$ and investigate some properties of those polynomials.

2. Degenerate Daehee polynomials

Let us assume that $\lambda, t \in \mathbb{C}_p$ with $|\lambda t|_p < p^{-\frac{1}{p-1}}$. We define the *degenerate high order Daehee polynomials* by the generating function as follows:

$$\begin{aligned}\sum_{n=0}^{\infty} D_{n,\lambda}^{(k)}(x)\frac{t^n}{n!} &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^{x_1+\cdots+x_k+x} d\mu_0(x_1)\cdots d\mu_0(x_k) \\ &= \left(\frac{\log\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)}{\log(1+\lambda t)^{\frac{1}{\lambda}}}\right)^k \left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^x.\end{aligned} \tag{2.1}$$

When $x = 0, k = 1$, $D_{n,\lambda} = D_{n,\lambda}(0)$ are called the *n-th degenerate Daehee numbers*.

It is well-known fact that the generating function of the Stirling number of the first kind is given by

$$(\log(1+t))^m = m!\sum_{l=m}^{\infty} S_1(l,m)\frac{t^l}{l!}, \text{ (see [3, 8, 14])}, \tag{2.2}$$

and the Stirling number of the second kind is defined by the generating function to be

$$(e^t-1)^n = \sum_{m=n}^{\infty} S_2(m,n)\frac{t^m}{m!}, \text{ (see [3, 14])}.$$

By (2.1) and (2.2), we observe that

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^{x_1+\cdots+x_k+x} d\mu_0(x_1)\cdots d\mu_0(x_k) \\ =&\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\binom{x_1+\cdots+x_k+x}{n} d\mu_0(x_1)\cdots d\mu_0(x_k)\lambda^{-n}\left(\log(1+\lambda t)\right)^n \\ =&\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\lambda^{n-l}S_1(n,l)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_k+x)_l d\mu_0(x_1)\cdots d\mu_0(x_k)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.3}$$

Thus, by (2.1) and (2.3), we obtain the following theorem.

Theorem 2.1. *For $n \geq 0$, we have*

$$D_{n,\lambda}^{(k)}(x) = \sum_{l=0}^{n}\lambda^{n-l}S_1(n,l)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_k+x)_l d\mu_0(x_1)\cdots d\mu_0(x_k).$$

By (1.5) and (1.6), we can note that

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{x_1+\cdots+x_k+x}d\mu_0(x_1)\cdots d\mu_0(x_k)\\
=&\sum_{n=0}^{\infty}\left(\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_k+x)_n d\mu_0(x_1)\cdots d\mu_0(x_k)\right)\frac{t^n}{n!}\\
=&\sum_{n=0}^{\infty}D_n^{(k)}(x)\frac{t^n}{n!}.
\end{aligned}\tag{2.4}$$

Therefore, by Theorem 2.1 and (2.4), we obtain the following corollary.

Corollary 2.2. *For $n \geq 0$, we have*

$$D_{n,\lambda}^{(k)}(x)=\sum_{l=0}^{n}\lambda^{n-l}S_1(n,l)D_l^{(k)}(x).$$

By replacing t by $\frac{1}{\lambda}(e^{\lambda t}-1)$ in (2.1), we get

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{x_1+\cdots+x_k+x}d\mu_0(x_1)\cdots d\mu_0(x_k)\\
=&\sum_{n=0}^{\infty}D_{n,\lambda}^{(k)}(x)\frac{1}{n!}\lambda^{-n}\left(e^{\lambda t}-1\right)^n\\
=&\sum_{n=0}^{\infty}D_{n,\lambda}^{(k)}(x)\frac{1}{n!}\lambda^{-n}\sum_{m=n}^{\infty}S_2(m,n)\frac{(\lambda t)^m}{m!}\\
=&\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}D_{m,\lambda}^{(k)}(x)\frac{1}{n!}\lambda^{n-m}S_2(n,m)\right)\frac{t^n}{n!}.
\end{aligned}\tag{2.5}$$

By (1.6) and (2.5), we obtain the following corollary.

Corollary 2.3. *For $n \geq 0$, we have*

$$D_n^{(k)}(x)=\sum_{m=0}^{n}D_{m,\lambda}^{(k)}(x)\frac{1}{n!}\lambda^{n-m}S_2(n,m).$$

By (1.7), we can derive the following equations easily:

$$\begin{aligned}
&\left(\frac{\log\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)}{\log(1+\lambda t)^{\frac{1}{\lambda}}}\right)^{k}\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^{x}\\
&=\sum_{n=0}^{\infty}B_n^{(n+k+1)}(x+1)\frac{1}{n!}\left(\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^{n}\\
&=\sum_{n=0}^{\infty}B_n^{(n+k+1)}(x+1)\lambda^{-n}\sum_{l=n}^{\infty}S_1(l,n)\frac{(\lambda t)^l}{l!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}B_m^{(m+k+1)}(x+1)\lambda^{n-m}S_1(n,m)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.6}$$

By (2.1) and (2.6), we obtain the following theorem.

Theorem 2.4. *For $n \geq 0$, we have*

$$D_{n,\lambda}^{(k)}(x)=\sum_{m=0}^{n}B_m^{(m+k+1)}(x+1)\lambda^{n-m}S_1(n,m).$$

We can observe that

$$\begin{aligned}
&\left(\frac{\log\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)}{\log(1+\lambda t)^{\frac{1}{\lambda}}}\right)^{k}\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^{x}\\
&=\left(\sum_{n=0}^{\infty}D_{n,\lambda}^{(k)}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\binom{x}{n}\left(\frac{1}{\lambda}\log(1+\lambda t)\right)^{n}\right)\\
&=\left(\sum_{n=0}^{\infty}D_{n,\lambda}^{(k)}\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}\left(\sum_{k=0}^{m}\lambda^{-k}(x)_kS_1(m,k)\right)\frac{(\lambda t)^m}{m!}\right)\\
&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=0}^{m}D_{n-m,\lambda}^{(k)}\binom{x}{l}\binom{n}{m}l!\lambda^{m-l}S_1(m,l)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.7}$$

Thus, by (2.1) and (2.7), we obtain the following theorem.

Theorem 2.5. *For $n \geq 0$, we have*

$$D_{n,\lambda}^{(k)}(x)=\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{x}{l}\binom{n}{m}l!\lambda^{m-l}S_1(m,l)D_{n-m,\lambda}^{(k)}. \tag{2.8}$$

It is well-known that the *high order Daehee polynomials of the second kind* are defined by the generating function to be

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{-(x_1+\cdots+x_k)+x}d\mu_0(x_1)\cdots d\mu_0(x_k)\\&=\left(\frac{\log(1+t)}{1-(1+t)^{-1}}\right)^k(1+t)^x\\&=\sum_{n=0}^{\infty}\widehat{D}_n^{(k)}(x)\frac{t^n}{n!},\end{aligned}\tag{2.9}$$

(see [6, 9, 10]).

From now on, we consider the *degenerate high order Daehee polynomials of the second kind* which are defined by the generating function to be

$$\begin{aligned}&\sum_{n=0}^{\infty}\widehat{D}_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}\\&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^{-x_1\cdots-x_k+x}d\mu_0(x_1)\cdots d\mu_0(x_k)\\&=\left(\frac{\log\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)}{1-\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^{-1}}\right)^k\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^x.\end{aligned}\tag{2.10}$$

By (2.10), we observe that

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^{-x_1\cdots-x_k+x}d\mu_0(x_1)\cdots d\mu_0(x_k)\\&=\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\binom{-x_1\cdots-x_k+x}{n}\left(\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^n d\mu_0(x_1)\cdots d\mu_0(x_k)\\&=\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\binom{-x_1\cdots-x_k+x}{n}d\mu_0(x_1)\cdots d\mu_0(x_k)\lambda^{-n}n!\sum_{m=n}^{\infty}S_1(m,n)\frac{(\lambda t)^m}{m!}\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\lambda^{n-m}S_1(n,m)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(-x_1\cdots-x_k+x)_m d\mu_0(x_1)\cdots d\mu_0(x_k)\right)\frac{t^n}{n!}.\end{aligned}\tag{2.11}$$

By (2.10) and (2.11), we obtain the following theorem.

Theorem 2.6. *For $n\geq 0$, we have*

$$\widehat{D}_{n,\lambda}^{(k)}(x)=\sum_{m=0}^{n}\lambda^{n-m}S_1(n,m)\widehat{D}_m^{(k)}(x).$$

By (2.10), we have

$$\begin{aligned}
&\left(\frac{\log\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)}{\log(1+\lambda t)^{\frac{1}{\lambda}}}\right)^k\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^x\\
&=\sum_{n=0}^{\infty}B_n^{(n+k+1)}(x+2)\frac{\left(\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^n}{n!}\\
&=\sum_{n=0}^{\infty}B_n^{(n+k+1)}(x+2)\lambda^{-n}\sum_{l=n}^{\infty}S_1(l,n)\frac{(\lambda t)^l}{l!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}B_m^{(m+k+1)}(x+2)\lambda^{n-m}S_1(n,m)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.12}$$

Therefore, by (2.12), we obtain the following theorem.

Theorem 2.7. *For $n \geq 0$, we have*

$$\widehat{D}_{n,\lambda}^{(k)}(x)=\sum_{m=0}^{n}B_m^{(m+k+1)}(x+2)\lambda^{n-m}S_1(n,m).$$

From (2.10) and (2.11), we have

$$\begin{aligned}
&\widehat{D}_{n,\lambda^{(k)}}(x)\\
&=\sum_{m=0}^{n}\lambda^{n-m}S_1(n,m)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(-x_1\cdots-x_k+x)_m d\mu_0(x_1)\cdots d\mu_0(x_k)\\
&=\sum_{m=0}^{n}\lambda^{n-m}S_1(n,m)(-1)^m\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1\cdots+x_k-x+m-1)_m d\mu_0(x_1)\cdots d\mu_0(x_k)\\
&=\sum_{m=0}^{n}\lambda^{n-m}S_1(n,m)(-1)^m m!\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\binom{x_1\cdots+x_k-x+m-1}{m}d\mu_0(x_1)\cdots d\mu_0(x_k)\\
&=\sum_{m=0}^{n}\lambda^{n-m}S_1(n,m)(-1)^m m!\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{l=0}^{m}\binom{m-1}{m-l}\binom{x_1\cdots+x_k-x}{l}d\mu_0(x_1)\cdots d\mu_0(x_k)\\
&=\sum_{m=0}^{n}\sum_{l=0}^{m}\lambda^{n-m}S_1(n,m)(-1)^m m!\binom{m-1}{m-l}\frac{1}{l!}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1\cdots+x_k-x)_l d\mu_0(x_1)\cdots d\mu_0(x_k)\\
&=\sum_{m=0}^{n}\sum_{l=1}^{m}\lambda^{n-m}S_1(n,m)(-1)^m m!\binom{m-1}{m-l}\frac{1}{l!}D_l^{(k)}(-x).
\end{aligned} \tag{2.13}$$

By (2.13), we obtain the following theorem.

Theorem 2.8. *For $n \geq 0$, we have*

$$\widehat{D}_{n,\lambda}^{k)}(x) = \sum_{m=0}^{n} \sum_{l=1}^{m} \frac{(-1)^m}{l!} m! \lambda^{n-m} S_1(n,m) \binom{m-1}{m-l} D_l^{(k)}(-x).$$

References

[1] A. Bayad and T. Kim, *Identities for the Bernoulli, the Euler and the Genocchi numbers and polynomials*, Adv. Stud. Contemp. Math. (Kyungshang), **20** (2010), no. 2, 247-253.

[2] L. Carlitz, *Degenerate Stirling, Bernoulli and Eulerian numbers*, Utilitas Math., **15** (1979), 51-88.

[3] L. Comtet, *Advanced Combinatorics*, Reidel, Dordrecht, 1974. http://dx.doi.org/10.1007/978-94-010-2196-8

[4] D. Ding and J. Yang, *Some identities related to the Apostol-Euler and Apostol-Bernoulli polynomials*, Adv. Stud. Contemp. Math. (Kyungshang), **20** (2010), no. 1, 7-21.

[5] S. Gaboury, R. Tremblay and B.-J. Fugere, *Some explicit formulas for certain new classes of Bernoulli, Euler and Genocchi polynomials*, Proc. Jangjeon Math. Soc., **17** (2014), no. 1, 115-123.

[6] D. S. Kim and T. Kim, *Daehee numbers and polynomials*, Appl. Math. Sci., **7** (2013), no. 117-120, 5969-5976. http://dx.doi.org/10.12988/ams.2013.39535

[7] D. S. Kim, T. Kim and D.V. Dolgy, *A note on degenerate Bernoulli numbers and polynomials associated with p-adic invariant integral on $\mathbb{Z}_p$*, Appl. Math. Comput., **259** (2015), 198-204. http://dx.doi.org/10.1016/j.amc.2015.02.068

[8] D. S. Kim, T. Kim and J. J. Seo, *Higher-order Daehee polynomials of the first kind with umbral calculus*, Adv. Stud. Contemp. Math., **24** (2014), no. 1, 5-18.

[9] D. S. Kim, T. Kim, S. H. Lee and J. J. Seo, *Higher-order Daehee numbers and polynomials*, Int. J. Math. Anal., **8** (2014), no. 6, 273-283. http://dx.doi.org/10.12988/ijma.2014.4118

[10] D. S. Kim, T. Kim, H. I. Kwon and J. J. Seo, *Daehee Polynomials with q-Parameter*, Adv. Studies Theor. Phys., **8** (2014), no. 13, 561-569. http://dx.doi.org/10.12988/astp.2014.4570

[11] T. Kim, *Barnes' type multiple degenerate Bernoulli and Euler polynomials*, Appl. Math. Comput., **258** (2015), 556-564. http://dx.doi.org/10.1016/j.amc.2015.02.040

[12] T. Kim, *Symmetry p-adic invariant integral on $\mathbb{Z}_p$ for Bernoulli and Euler polynomials*, J. Difference Equ. Appl., **14** (2008), no. 12, 1267-1277. http://dx.doi.org/10.1080/10236190801943220

[13] T. Kim, *Multiple p-adic L-function*, Russ. J. Math. Phys., **13** (2006), no. 2, 151-157. http://dx.doi.org/10.1134/s1061920806020038

[14] S. Roman, *The umbral calculus*, Dover, New York, 2005.

[15] L. C. Washington, *Introduction to cyclotomic fields, Second edition. Graduate Texts in Mathematics, 83*, Springer-Verlag, New York, 1997. http://dx.doi.org/10.1007/978-1-4612-1934-7

Received: June 17, 2015; Published: June 27, 2015