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Continuous Dependence of the Solution of Itô Stochastic Differential Equation with Nonlocal Conditions

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Abstract

In this paper we are concerned with a nonlocal problem of a stochastic differential equation of Itô type. The solution contains both of Riemann (or Lebesgue) and Itô integrals in the mean square sense, so we study the existence of a unique mean square continuous solution and its continuous dependence on the random data X_0 and on the (non-random data) coefficients of the nonlocal condition a_k. Also, a stochastic differential equation with the integral condition will be considered.

Keywords: Riemann integral, Itô integral, Brownian motion, nonlocal condition, unique mean square solution, continuous dependence, random data, non-random data, integral condition

1 Introduction

Stochastic differential equations have been extensively studied by several authors specially stochastic differential equations of Itô's type, they studied Itô's integral in mean square sense as in ([19]) and in almost certain sense as in ([2]), properties of Brownian motion (or a Wiener process) as a formal derivative of the Gausian white noise occupied much attention of authors. A Brownian motion $W(t), t \in R$, is defined as a stochastic process such that

$$W(0) = 0, \ E(W(t)) = 0, \ \ E(W(t))^2 = t$$

and $[W(t_1) - W(t_2)]$ is a Gaussian random variable for all $t_1, t_2 \in R$.
The reader is referred to ([1]-[3]), ([7]) and ([12]-[21]) and references therein.
Also problems with nonlocal conditions have been heavily studied by several authors in the last decades in the ordinary differential equations. The reader is referred to ([4]-[6]) and ([8]-[11]), and references therein.
Here we are concerned with the stochastic differential equation of Itô's type

$$dX(t) = f(t, X(t))dt + g(t, X(t))dW(t), \quad t \in (0, T] \tag{1}$$

with the nonlocal random initial condition

$$X(0) \ + \ \sum_{k=1}^{n} a_k \ X(\tau_k) \ = \ X_0, \quad a_k \ > 0 \ , \ \tau_k \ \in \ (0, \ T), \tag{2}$$

where X_0 is a second order random variable independent of the Brownian motion (or Wiener process) $W(t)$ and a_k are positive real numbers.
The existence of a unique mean square continuous solution will be studied. The continuous dependence on the random data X_0 and the non-random data a_k will be established. The problem (1) with the mean square Riemann-Steltjes integral condition

$$X(0) + \int_0^T X(s)dv(s) = X_0. \tag{3}$$

will be considered.

2 Existence and uniqueness

Let $I = [0, T]$ and $C = C(I, L_2(\Omega))$ be the class of all mean square continuous second order stochastic process with the norm

$$\| X \|_C = \sup_{t \in [0,T]} \| X(t) \|_2 = \sup_{t \in [0,T]} \sqrt{E(X(t))^2}.$$

Throughout the paper we assume that the following assumptions hold

(H1) The functions $f : [0, T] \times L_2(\Omega) \to L_2(\Omega)$ and $g : [0, T] \times L_2(\Omega) \to L_2(\Omega)$ are mean square continuous, and there exists a positive real numbers r_1 and r_2 such that

$$\sup_{t \in [0,T]} | f(t, 0) | \le r_1, \qquad \sup_{t \in [0,T]} | g(t, 0) | \le r_2.$$

(H2) There exists an integrable functions $k_1 : [0,T] \to R^+$ and $k_2 : [0,T] \to R^+$, where

$$\sup_{t\in[0,T]} \int_0^t k_1(s)ds \le m_1, \qquad \sup_{t\in[0,T]} \int_0^t k_2^2(s)ds \le m_2^2,$$

such that the function f and g satisfy the mean square Lipschitz condition

$$\| f(t,X_1(t)) - f(t,X_2(t)) \|_2 \le k_1(t) \| X_1(t) - X_2(t) \|_2$$

and

$$\| g(t,X_1(t)) - g(t,X_2(t)) \|_2 \le k_2(t) \| X_1(t) - X_2(t) \|_2 .$$

Now we have the following lemma.

Lemma 2.1 *The solution of the nonlocal stochastic problem (1)-(2) can be expressed by the stochastic integral equation*

$$\begin{aligned} X(t) &= a\left(X_0 - \sum_{k=1}^n a_k \int_0^{\tau_k} f(s,X(s))ds - \sum_{k=1}^n a_k \int_0^{\tau_k} g(s,X(s))dW(s)\right) \\ &+ \int_0^t f(s,X(s))ds + \int_0^t g(s,X(s))dW(s), \end{aligned} \tag{4}$$

where $a = \left(1 + \sum_{k=1}^n a_k\right)^{-1}$.

Proof. Integrating equation (1), we obtain

$$X(t) = X(0) + \int_0^t f(s,X(s))ds + \int_0^t g(s,X(s))dW(s),$$

then

$$\sum_{k=1}^n a_k X(\tau_k) = \sum_{k=1}^n a_k X(0) + \sum_{k=1}^n a_k \int_0^{\tau_k} f(s,X(s))ds + \sum_{k=1}^n \int_0^{\tau_k} g(s,X(s))dW(s)$$

$$X_0 - X(0) = \sum_{k=1}^n a_k X(0) + \sum_{k=1}^n a_k \int_0^{\tau_k} f(s,X(s))ds + \sum_{k=1}^n \int_0^{\tau_k} g(s,X(s))dW(s)$$

and

$$\left(1 + \sum_{k=1}^n a_k\right) X(0) = X_0 - \sum_{k=1}^n a_k \int_0^{\tau_k} f(s,X(s))ds - \sum_{k=1}^n \int_0^{\tau_k} g(s,X(s))dW(s),$$

then

$$X(0)=\left(1+\sum_{k=1}^{n}a_k\right)^{-1}\left(X_0-\sum_{k=1}^{n}a_k\int_0^{\tau_k}f(s,X(s))ds-\sum_{k=1}^{n}a_k\int_0^{\tau_k}g(s,X(s))dW(s)\right).$$

Hence

$$\begin{aligned}X(t) &= a\left(X_0-\sum_{k=1}^{n}a_k\int_0^{\tau_k}f(s,X(s))ds-\sum_{k=1}^{n}a_k\int_0^{\tau_k}g(s,X(s))dW(s)\right)\\ &+ \int_0^{t}f(s,X(s))ds+\int_0^{t}g(s,X(s))dW(s),\end{aligned}$$

where $a=\left(1+\sum_{k=1}^{n}a_k\right)^{-1}$. ■

Now define the mapping

$$\begin{aligned}AX(t) &= a\left(X_0-\sum_{k=1}^{n}a_k\int_0^{\tau_k}f(s,X(s))ds-\sum_{k=1}^{n}a_k\int_0^{\tau_k}g(s,X(s))dW(s)\right)\\ &+ \int_0^{t}f(s,X(s))ds+\int_0^{t}g(s,X(s))dW(s).\end{aligned}$$

Then we can prove the following lemma.

Lemma 2.2 $A : C \to C$.

Proof. Let $X \in C$, t_1 , $t_2 \in [0,T]$ such that $\mid t_2-t_1 \mid < \delta$, then

$$A\ X(t_2)-A\ X(t_1)=\int_{t_1}^{t_2}f(s,X(s))ds+\int_{t_1}^{t_2}g(s,X(s))dW(s).$$

From assumption (H2) we have

$$\| f(t,X(t)) \|_2 - \mid f(t,0) \mid \leq \| f(t,X(t))-f(t,0) \|_2 \leq k_1(t) \parallel X(t) \parallel_2,$$

then we have

$$\| f(t,X(t)) \|_2 \leq k_1(t) \ \ \| X \|_C + r_1$$

and similarly,

$$\| g(t,X(t)) \|_2 \leq k_2(t) \ \ \| X \|_C + r_2,$$

then we have

$$\| A\ X(t_2)-A\ X(t_1) \|_2 \leq \left\|\int_{t_1}^{t_2}f(s,X(s))ds\right\|_2+\left\|\int_{t_1}^{t_2}g(s,X(s))dW(s)\right\|_2.$$

Now

$$\left\|\int_{t_1}^{t_2} f(s,X(s))ds\right\|_2 \leq \int_{t_1}^{t_2} ||f(s,X(s))||_2\, ds \leq \int_{t_1}^{t_2} [k_1(s) \ \parallel X \parallel_C + r_1] ds$$

$$\leq \ \parallel X \parallel_C \int_{t_1}^{t_2} k_1(s)\ ds + r_1(t_2 - t_1)$$

and

$$\left\|\int_{t_1}^{t_2} g(s,X(s))dW(s)\right\|_2^2 = \int_{t_1}^{t_2} ||g(s,X(s))||_2^2\, ds \leq \int_{t_1}^{t_2} [k_2(s) \ \parallel X \parallel_C + g(s,0)]^2 ds$$

$$\leq \ 2 \parallel X \parallel_C^2 \int_{t_1}^{t_2} k_2^2(s)\ ds + 2\int_{t_1}^{t_2} \mid g(s,0) \mid^2\ ds$$

$$\leq \ 2 \parallel X \parallel_C^2 \int_{t_1}^{t_2} k_2^2(s)\ ds + 2r_2^2(t_2 - t_1).$$

So,

$$\parallel A\ X(t_2) - A\ X(t_1) \parallel_2 \leq$$

$$\parallel X \parallel_C \int_{t_1}^{t_2} k_1(s)\ ds + r_1(t_2 - t_1) + \parallel X \parallel_C \sqrt{2\int_{t_1}^{t_2} k_2^2(s)\ ds} + r_2\sqrt{2(t_2 - t_1)},$$

which proves that $A\ :\ C\ \rightarrow\ C$.■

For the existence of a unique mean square continuous solution $X \in C$ of the problem (1)-(2), we have the following theorem.

Theorem 2.1 *Let the assumptions (H1)-(H2) be satisfied. If* $2(m_1 + m_2) < 1$, *then the problem (1)-(2) has a unique solution* $X \in C$.

Proof. Let X and $X^* \in C$, then

$$\parallel AX(t) - AX^*(t) \parallel_2 \leq \left\|\int_0^t [f(s,X(s)) - f(s,X^*(s))]ds - a\sum_{k=1}^n a_k \int_0^{\tau_k} [f(s,X(s)) - f(s,X^*(s))]ds\right\|_2$$

$$+ \left\|\int_0^t [g(s,X(s)) - g(s,X^*(s))]dW(s) - a\sum_{k=1}^n a_k \int_0^{\tau_k} [g(s,X(s)) - g(s,X^*(s))]dW(s)\right\|_2,$$

we have

$$\left\|\int_0^t [f(s,X(s)) - f(s,X^*(s))]ds\right\|_2 \leq \ \parallel X - X^* \parallel_C \int_0^t k_1(s)ds \leq m_1 \parallel X - X^* \parallel_C$$

and

$$\left\|\int_0^t [g(s,X(s)) - g(s,X^*(s))]dW(s)\right\|_2^2 = \int_0^t ||g(s,X(s)) - g(s,X^*(s))||_2^2\, ds$$
$$\leq \| X - X^* \|_C^2 \int_0^t | k_2(s) |^2 \, ds \leq m_2^2 \| X - X^* \|_C^2 .$$

Hence

$$\| AX - AX^* \|_C \leq 2(m_1 + m_2) \| X - X^* \|_C .$$

If $2(m_1+m_2) < 1$, then A is contraction and there exists a unique solution $X \in C$ of the nonlocal stochastic problem (1)-(2), [3]. This solution is given by (4).■

3 Continuous dependence

Consider the stochastic differential equation (1) with the nonlocal condition

$$X(0) + \sum_{k=1}^{n} a_k X(\tau_k) = \tilde{X}_0, \qquad \tau_k \in (0, T) \tag{5}$$

Definition 3.1 *The solution $X \in C$ of the nonlocal problem (1),(2) is continuously dependent (on the data X_0) if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\| X_0 - \tilde{X}_0 \|_2 \leq \delta$ implies that $\| X - \tilde{X} \|_C \leq \epsilon$*

Here, we study the continuous dependence (on the random data X_0) of the solution of the stochastic differential equation (1) and (2).

Theorem 3.2 *Let the assumptions (H1)-(H2) be satisfied. Then the solution of the nonlocal problem (1)-(2) is continuously dependent on the random data X_0.*

Proof. Let $X(t)$ as defined in equation (4) be the solution of the nonlocal problem (1)-(2) and

$$\tilde{X}(t) = a\left(\tilde{X}_0 - \sum_{k=1}^{n} a_k \int_0^{\tau_k} f(s,\tilde{X}(s))ds - \sum_{k=1}^{n} a_k \int_0^{\tau_k} g(s,\tilde{X}(s))dW(s)\right)$$
$$+ \int_0^t f(s,\tilde{X}(s))ds + \int_0^t g(s,\tilde{X}(s))dW(s)$$

be the solution of the nonlocal problem (1) and (5). Then

$$X(t) - \tilde{X}(t) =$$

$$a[X_0-\tilde{X}_0]-a\sum_{k=1}^{n}a_k\int_0^{\tau_k}[f(s,X(s))-f(s,\tilde{X}(s))]ds$$
$$-a\sum_{k=1}^{n}a_k\int_0^{\tau_k}[g(s,X(s))-g(s,\tilde{X}(s))]dW(s)$$
$$+\int_0^t[f(s,X(s))-f(s,\tilde{X}(s))]ds+\int_0^t[g(s,X(s))-g(s,\tilde{X}(s))]dW(s).$$

Using our assumptions, we get

$$\begin{aligned}\| X(t)-\tilde{X}(t)\|_2 &\leq a\| X_0-\tilde{X}_0\|_2 + a\sum_{k=1}^{n}a_k\int_0^{\tau_k}\| f(s,X(s))-f(s,\tilde{X}(s))\|_2\, ds\\ &+ a\sum_{k=1}^{n}a_k\left\|\int_0^{\tau_k}[g(s,X(s))-g(s,\tilde{X}(s))]dW(s)\right\|_2\\ &+ \int_0^t\| f(s,X(s))-f(s,\tilde{X}(s))\|_2\, ds+\left\|\int_0^t[g(s,X(s))-g(s,\tilde{X}(s))]dW(s)\right\|_2\end{aligned}$$

then we can get

$$\begin{aligned}\| X-\tilde{X}\|_C &\leq a\delta+a\sum_{k=1}^{n}a_k m_1\| X-\tilde{X}\|_C + a\sum_{k=1}^{n}a_k m_2\| X-\tilde{X}\|_C\\ &+ m_1\| X-\tilde{X}\|_C + m_2\| X-\tilde{X}\|_C\\ &\leq a\delta+2(m_1+m_2)\| X-\tilde{X}\|_C\end{aligned}$$

Hence

$$\| X-\tilde{X}\|_C \leq \frac{a\delta}{1-2(m_1+m_2)}.$$

This complete the proof. ■

Now consider the stochastic differential equation (1) with the nonlocal condition

$$X(0)+\sum_{k=1}^{n}\tilde{a}_k\, X(\tau_k)=X_0,\qquad \tau_k\in(0,\ T) \tag{6}$$

Definition 3.2 *The solution* $X\in C$ *of the nonlocal problem (1)-(2) is continuously dependent (on the data* a_k*) if* $\forall\epsilon>0$, $\exists\delta>0$ *such that* $|a_k-\tilde{a}_k|\leq\delta$ *implies that* $\| X-\tilde{X}\|_C\leq\epsilon$

Here, we study the continuous dependence (on the coefficient a_k of the nonlocal condition) of the solution of the stochastic differential equation (1) and (2).

Theorem 3.3 *Let the assumptions (H1)-(H2) be satisfied. Then the solution of the nonlocal problem (1)-(2) is continuously dependent on the coefficients* a_k *of the nonlocal condition.*

Proof. Let $X(t)$ as defined in equation (4) be the solution of the nonlocal problem (1)-(2) and

$$\begin{aligned}\tilde{X}(t) &= \tilde{a}\left(X_0 - \sum_{k=1}^{n}\tilde{a}_k\int_0^{\tau_k} f(s,\tilde{X}(s))ds - \sum_{k=1}^{n}\tilde{a}_k\int_0^{\tau_k} g(s,\tilde{X}(s))dW(s)\right)\\ &+ \int_0^t f(s,\tilde{X}(s))ds + \int_0^t g(s,\tilde{X}(s))dW(s)\end{aligned}$$

be the solution of the nonlocal problem (1) and (6). Then

$$\begin{aligned}X(t)-\tilde{X}(t) &= [a-\tilde{a}]X_0 + \int_0^t [f(s,X(s)) - f(s,\tilde{X}(s))]ds + \int_0^t [g(s,X(s)) - g(s,\tilde{X}(s))]ds\\ &- a\sum_{k=1}^{n} a_k\int_0^{\tau_k} f(s,X(s))ds + \tilde{a}\sum_{k=1}^{n}\tilde{a}_k\int_0^{\tau_k} f(s,\tilde{X}(s))ds\\ &- a\sum_{k=1}^{n} a_k\int_0^{\tau_k} g(s,X(s))ds + \tilde{a}\sum_{k=1}^{n}\tilde{a}_k\int_0^{\tau_k} g(s,\tilde{X}(s))ds.\end{aligned}$$

Now

$$|\, a-\tilde{a}\,| = \left|\frac{1}{1+\sum_{k=1}^{n} a_k} - \frac{1}{1+\sum_{k=1}^{n}\tilde{a}_k}\right| = \left|\frac{\sum_{k=1}^{n}(\tilde{a}_k - a_k)}{\left(1+\sum_{k=1}^{n} a_k\right)\left(1+\sum_{k=1}^{n}\tilde{a}_k\right)}\right| \leq \left|\sum_{k=1}^{n}(\tilde{a}_k - a_k)\right| \leq n\delta,$$

and

$$\tilde{a}\sum_{k=1}^{n}\tilde{a}_k\int_0^{\tau_k} f(s,\tilde{X}(s))ds - a\sum_{k=1}^{n}a_k\int_0^{\tau_k} f(s,X(s))ds = \tilde{a}\left(1+\sum_{k=1}^{n}\tilde{a}_k\right)\int_0^{\tau_k} f(s,\tilde{X}(s))ds$$

$$\begin{aligned}
&- \; a\left(1+\sum_{k=1}^{n}a_k\right)\int_0^{\tau_k} f(s,X(s))ds - \tilde{a}\int_0^{\tau_k} f(s,\tilde{X}(s))ds + a\int_0^{\tau_k} f(s,X(s))ds \\
&= \; \tilde{a}(\tilde{a}^{-1})\int_0^{\tau_k} f(s,\tilde{X}(s))ds - a(a^{-1})\int_0^{\tau_k} f(s,X(s))ds - \tilde{a}\int_0^{\tau_k} f(s,\tilde{X}(s))ds + a\int_0^{\tau_k} f(s,X(s))ds \\
&= \; -\int_0^{\tau_k}[f(s,X(s))-f(s,\tilde{X}(s))]ds + a\int_0^{\tau_k} f(s,X(s))ds - \tilde{a}\int_0^{\tau_k} f(s,\tilde{X}(s))ds \\
&- \; \tilde{a}\int_0^{\tau_k} f(s,X(s))ds + \tilde{a}\int_0^{\tau_k} f(s,X(s))ds \\
&= \; -\int_0^{\tau_k}[f(s,X(s))-f(s,\tilde{X}(s))]ds + [a-\tilde{a}]\int_0^{\tau_k} f(s,X(s))ds + \tilde{a}\int_0^{\tau_k}[f(s,X(s))-f(s,\tilde{X}(s))]ds,
\end{aligned}$$

similarly

$$\tilde{a}\sum_{k=1}^{n}\tilde{a}_k\int_0^{\tau_k} g(s,\tilde{X}(s))ds - a\sum_{k=1}^{n}a_k\int_0^{\tau_k} g(s,X(s))ds$$

$$= \; -\int_0^{\tau_k}[g(s,X(s))-g(s,\tilde{X}(s))]ds + [a-\tilde{a}]\int_0^{\tau_k} g(s,X(s))ds + \tilde{a}\int_0^{\tau_k}[g(s,X(s))-g(s,\tilde{X}(s))]ds.$$

Then,

$$\| X(t)-\tilde{X}(t)\|_2 \leq$$

$$n\delta \| X_0\|_2 + \int_{\tau_k}^{t} \| f(s,X(s))-f(s,\tilde{X}(s))\|_2 \, ds + \left\| \int_{\tau_k}^{t}[g(s,X(s))-g(s,\tilde{X}(s))]ds \right\|_2$$

$$+n\delta\int_0^{\tau_k} \| f(s,X(s))\|_2 \, ds + \tilde{a}\int_0^{\tau_k} \| f(s,X(s))-f(s,\tilde{X}(s))\|_2 \, ds$$

$$+n\delta\left\|\int_0^{\tau_k} g(s,X(s))ds\right\|_2 + \tilde{a}\left\|\int_0^{\tau_k}[g(s,X(s))-g(s,\tilde{X}(s))]ds\right\|_2 .$$

Using our assumptions we get

$$\begin{aligned}
\| X-\tilde{X}\|_C \;\leq\; & n\delta \| X_0\|_2 + m_1 \| X-\tilde{X}\|_C + m_2 \| X-\tilde{X}\|_C + n\delta(\| X\|_C \, m_1 + r_1 T) \\
+ \; & \tilde{a}m_1 \| X-\tilde{X}\|_C + n\delta\sqrt{2}(\| X\|_C \, m_2 + r_2\sqrt{T}) + \tilde{a}m_2 \| X-\tilde{X}\|_C,
\end{aligned}$$

and

$$\begin{aligned}\| X-\tilde{X} \|_C &\leq n\delta\Big[\| X_0 \|_C + m_1 \| X \|_C + r_1 T + \sqrt{2}(\| X \|_C\, m_2 + r_2\sqrt{T})\Big] \\ &+ [m_1 + \tilde{a}m_1 + m_2 + \tilde{a}m_2] \| X - \tilde{X} \|_C \\ &\leq n\delta\Big[\| X_0 \|_C + m_1 \| X \|_C + r_1 T + \sqrt{2}(\| X \|_C\, m_2 + r_2\sqrt{T})\Big] + 2(m_1+m_2) \| X - \tilde{X} \|_C .\end{aligned}$$

Hence

$$\| X - \widetilde{X} \|_C \leq \frac{n\delta\Big[\| X_0 \|_C + m_1 \| X \|_C + r_1 T + \sqrt{2}(\| X \|_C\, m_2 + r_2\sqrt{T})\Big]}{1 - 2(m_1 + m_2)}.$$

This complete the proof. ■

4 Nonlocal integral condition

Let $v : [0,T] \to [0,T]$ be nondecreasing function such that $a_k = v(t_k) - v(t_{k-1}),\ \tau_k \in (t_{k-1}, t_k)$, where $(0 < t_1 < t_2 < t_3 < .... < T)$.
Then, the nonlocal condition (2) will be in the form

$$X(0) + \sum_{k=1}^{n} X(\tau_k)\ (v(t_k) - v(t_{k-1})) = X_0.$$

From the mean square continuity of the solution of the nonlocal problem (1)-(2), we obtain from [19]

$$\underset{n\to\infty}{l.i.m} \sum_{k=1}^{n} X(\tau_k)\ (v(t_k) - v(t_{k-1})) = \int_0^T X(s)dv(s),$$

that is, the nonlocal conditions (2) is transformed to the mean square Riemann-Steltjes integral condition

$$X(0) + \int_0^T X(s)dv(s) = X_0,$$

Now, we have the following theorem.

Theorem 4.4 *Let the assumptions (H1)-(H2) be satisfied, then the stochastic differential equation (1) with the nonlocal integral condition (3) has a unique solution represented in the form*

$$\begin{aligned}X(t) &= a^\star\left(X_0 - \int_0^T\int_0^s f(\theta, X(\theta))d\theta dv(s) - \int_0^T\int_0^s g(\theta, X(\theta))dW(\theta)dv(s)\right) \\ &+ \int_0^t f(\theta, X(\theta))d\theta + \int_0^t g(\theta, X(\theta))dW(\theta).\end{aligned}$$

where $a^\star = (1 + v(T) - v(0))^{-1}$.

Proof. Taking the limit of equation (4) we get the proof. ■

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