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Numerical Method for Unsteady Fluid Structure Interaction Problem

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Abstract

In this paper, we combine finite difference method and finite element method to solve coupled problem. The incompressible viscous fluid is governed by Navier-Stokes equations and the elastic beam is modelled by the Euler-Bernoulli equation. The numerical method is based on leap-flog scheme.

Mathematics Subject Classification: 64F10, 65N06, 65N30, 65N12

Keywords: Fluid-structure interaction, Euler-Bernouilli equation, Formulation ALE, Navier-Stokes equation, finite element method, finite difference method, leap-flog scheme

1 Introduction

This article is motivated by the numeric simulation of fluid structure interaction phenomenon. The coupled system is made of viscous newtonien and incompressible fluid gouverned by the Navier-Stokes equation completed by the beam equation modelling the displacement of the structure as well as limits and initial conditions required: it's a heiglly coupled model for the structure

mouvement because is influenced by the fluid efforts on the structure and vice versa. A great number of situations utilize phenomenon of interaction fluid-structure. Namely, the flow around a boat, blood flow in the arteries, flow around plane wings, among the stridied flow models. The one we are interested most in this stydy is the blood flow in the arteries. This work is an extension of the method developed in [2]. In this present work we'll focus on a numeric scheme of coupling between a deformed structure and an imcompressible viscous fluid for the calculation of parameters so as the pressure of the fluid, its velocity as well as the displacement of the structure.
Amongst the computational methods for fluid structure interaction problem, we cite the fixed point method, the Newton method, the Quasi-Newton method, the fictitious domain method. Thus, this paper aims at showing that, we can combine the finite difference method, the finite element method to solve fluid structure interaction unsteady problem . On the one hand, we use finite difference method to approximate the structure model in order to have a linear dynamic systems, on the other hand, we solve the Navier-Stokes equation by the finite element method. Moreover, we will be intruduced the leap-flog scheme to compute the displacement of the structure. Thus, the velocity v and the pressure p of the fluid are done in the deformed domain.

2 Position of problem

2.1 Domain fluid

We consider $\Omega_F^u \subset \mathbb{R}^2$ the domain occupied by the fluid and Γ_2 the interface between the fluid and the elastic structure.

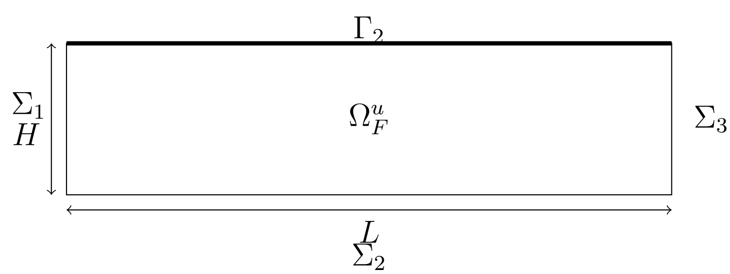

Figure 1: Computational domain

Where, the border $\partial\Omega_F^u = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Gamma_2$.

- Σ_1 is the inflow
- Σ_2 is a rigid border
- Σ_3 is the outflow
- L is the domain length
- H is the domain height
- u is the displacement of the structure

2.2 Fluid properties

The fluid is considered to be newtonian, incompressible, viscous and its state is described by the velocity $v = (v_1, v_2)$ and the pressure p . The balance equations are

$$
\begin{aligned}
\rho^f(\frac{\partial v}{\partial t} + (v.\nabla)v) - \mu\Delta v + \nabla p &= f^F, \text{ in } \Omega_F^u, t > 0 && (1)\\
\nabla.v &= 0, \text{ in } \Omega_F^u, t > 0 && (2)\\
v(t=0) &= v_0, \text{ in } \Omega_F^u, && (3)\\
-pI_d n + \mu\frac{\partial v}{\partial n} &= p_{in} I_d n, \text{ on } \Sigma_1, t > 0 && (4)\\
v_1 = v_2 &= 0, \text{ on } \Sigma_2, t > 0 && (5)\\
-pI_d n + \mu\frac{\partial v}{\partial n} &= 0, \text{ on } \Sigma_3, t > 0 && (6)\\
v_1 &= 0 \text{ on } \Gamma_2, t > 0 && (7)\\
v_2 &= \frac{\partial u}{\partial t} \text{ on } \Gamma_2, t > 0 && (8)
\end{aligned}
$$

- ρ^f: is the density of the fluid
- μ: the fluid viscosity
- I_d: the identity matrix
- v_1: the first component of v
- v_2: the second component of v
- $f^F = (f_1^F, f_2^F)$: the volume force of the fluid
- n: is a unit normal vector
- v_0: the initial velocity

2.3 Structure properties

The structure is assumed by elastic beam. We note $u : [0, L] \times \mathbb{R}^{+} \longrightarrow \mathbb{R}$ the displacement of the structure, it is modelled by the Euler-Bernouilli equation

$$\rho^{s}\frac{\partial^{2}u}{\partial t^{2}}(x,t) + D\frac{\partial^{4}u}{\partial x^{4}}(x,t) = p(x, H + u(x,t)),\ \forall x \in]0, L[,\ t > 0 \qquad (9)$$

with the boundary conditions,

$$\begin{aligned} u(0,t) = u(L,t) &= 0 \qquad \forall t > 0 && (10)\\ \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) &= 0 \qquad \forall t > 0 && (11)\\ u(x,0) &= u_0(x) \qquad \forall x \in]0, L[&& (12)\\ \frac{\partial u}{\partial t}(x,0) &= u_1(x) \qquad \forall x \in]0, L[&& (13) \end{aligned}$$

Where,

- $D = \frac{E \times h^3}{12(1-\nu^2)}$

- E is the Young modulus

- h elastic structure thickness

- ν the Poisson's coefficient

- ρ^s is the structure density

Remark: In equation (9) we assume that only the pressure force is acting on the interface and also u is the transversal displacement [8].

3 Coupled problem

The coupled problem is to find (u, v, p) such that:

$$\begin{cases} \rho^s \frac{\partial^2 u}{\partial t^2}(x,t) + D\frac{\partial^4 u}{\partial x^4}(x,t) = p(x, H + u(x,t)),\ \forall x \in]0, L[,\ t > 0 \\ u(0,t) = u(L,t) = 0 \qquad \forall t > 0 \\ \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0 \qquad \forall t > 0 \\ u(x,0) = u_0(x) \qquad \forall x \in]0, L[\\ \frac{\partial u}{\partial t}(x,0) = u_1(x) \qquad \forall x \in]0, L[\\ \rho^f(\frac{\partial v}{\partial t} + (v.\nabla)v) - \mu\Delta v + \nabla p = f^F, \text{ in } \Omega_F^u, t > 0 \\ \nabla.v = 0, \text{ in } \Omega_F^u, t > 0 \\ v(t=0) = v_0, \text{ in } \Omega_F^u \\ -pI_d n + \mu\frac{\partial v}{\partial n} = p_{in} I_d n, \text{ on } \Sigma_1, t > 0 \\ v_1 = v_2 = 0, \text{ on } \Sigma_2, t > 0 \\ -pI_d n + \mu\frac{\partial v}{\partial n} = 0, \text{ on } \Sigma_3, t > 0 \\ v_1 = 0 \text{ on } \Gamma_2, t > 0 \\ v_2 = \frac{\partial u}{\partial t} \text{ on } \Gamma_2, t > 0 \end{cases}$$

In order to solve this coupled problem, we transform its continuous problem into a discreet problem by using finite difference method and finite element method.

Assumption: We consider u as a small displacement.
Thus, the Taylor formula gives

$$p(x, H + u(x,t)) \approx p(x,H) + u(x,t)\frac{\partial p(x,H)}{\partial y}, \tag{14}$$

the equation (9) becomes:

$$\rho^s \frac{\partial^2 u}{\partial t^2}(x,t) + D\frac{\partial^4 u}{\partial x^4}(x,t) - u(x,t)\frac{\partial p(x,H)}{\partial y} = p(x,H), \tag{15}$$

we pose $\alpha(x) = -\frac{\partial p(x,H)}{\partial y}$, finally we have,

$$\rho^s \frac{\partial^2 u}{\partial t^2}(x,t) + D\frac{\partial^4 u}{\partial x^4}(x,t) + \alpha(x)u(x,t) = p(x,H). \tag{16}$$

To discretize the domain $]0, L[\times\mathbb{R}^+$, we introduce a space step $\Delta x = \frac{L}{N+1}$ and a time step $\Delta t > 0$. We define the nodes of a regular mesh

$$(x_i, t_n) = (i\Delta x, n\Delta t) for\, i \in \{0, 1, \ldots, N+1\}, n \geq 0$$

We denote by u_i^n the value of the discrete solution at (x_i, t_n).
To solve the equetion (15), we approximate the second time derivative by

$$\frac{\partial^2 u}{\partial t^2}(t_n, x_i) \approx \frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta t^2},$$

and $\dfrac{\partial^4 u}{\partial x^4}(x_i t_n)$ by

$$\frac{\partial^4 u}{\partial x^4}(x_i, t_n) \approx \frac{u_{i-2}^n - 4u_{i-1}^n + 6u_i^n - 4u_{i+1}^n + u_{i+2}^n}{\Delta x^4}$$

The initial data is discretized by

$$u_i^0 = u_0(x_i) \qquad for\, i \in \{0, 1, \ldots, N+1\}$$

We must also discretize the boundary conditions . For (11), a centred formula gives

$$u_1^n = u_{-1}^n \; et \; u_{N+2}^n = u_N^n \tag{17}$$

and the boundary conditions $u(0,t) = u(L,t) = 0$ become

$$u_0^n = u_{N+1}^n = 0. \tag{18}$$

Moreover the discretization of (13) give

$$u_i^1 \approx u_i^0 + \Delta t u_1(x_i)$$

we rewrite the equation (16) in the discreet form

$$\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta t^2} + D\frac{u_{i-2}^n - 4u_{i-1}^n + 6u_i^n - 4u_{i+1}^n + u_{i+2}^n}{\Delta x^4} + \alpha_i u_i^n = P(x_i, H) \tag{19}$$

for $n \geq 0$ et $i = 1, 2, 3 \ldots N$.
Then, the continuous problem becomes the following algebraic equation

$$U^{n+1} + U^{n-1} + AU^n = P$$

such as

$$A = B + C$$

Where

$$U^{n+1} = \begin{pmatrix} u_1^{n+1} \\ u_2^{n+1} \\ u_3^{n+1} \\ \vdots \\ u_N^{n+1} \end{pmatrix} \qquad U^{n-1} = \begin{pmatrix} u_1^{n-1} \\ u_2^{n-1} \\ u_3^{n-1} \\ \vdots \\ u_N^{n-1} \end{pmatrix} \qquad U^n = \begin{pmatrix} u_1^n \\ u_2^n \\ u_3^n \\ \vdots \\ u_N^n \end{pmatrix} \quad P = \begin{pmatrix} \Delta t^2 p\,(x_1, H) \\ \Delta t^2 p\,(x_2, H) \\ \Delta t^2 p\,(x_3, H) \\ \vdots \\ \Delta t^2 p\,(x_N, H) \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{7D\Delta t^2}{\Delta x^4} & \frac{-4D\Delta t^2}{\Delta x^4} & \frac{D\Delta t^2}{\Delta x^4} & 0 & 0 & \dots & 0 \\ \frac{-4D\Delta t^2}{\Delta x^4} & \frac{6D\Delta t^2}{\Delta x^4} & \frac{-4D\Delta t^2}{\Delta x^4} & \frac{D\Delta t^2}{\Delta x^4} & 0 & \dots & 0 \\ \frac{D\Delta t^2}{\Delta x^4} & \frac{-4D\Delta t^2}{\Delta x^4} & \frac{6D\Delta t^2}{\Delta x^4} & \frac{-4D\Delta t^2}{\Delta x^4} & \frac{D\Delta t^2}{\Delta x^4} & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \frac{D\Delta t^2}{\Delta x^4} & \frac{-4D\Delta t^2}{\Delta x^4} & \frac{6D\Delta t^2}{\Delta x^4} & \frac{-4D\Delta t^2}{\Delta x^4} & \frac{D\Delta t^2}{\Delta x^4} \\ 0 & \dots & 0 & \frac{D\Delta t^2}{\Delta x^4} & \frac{-4D\Delta t^2}{\Delta x^4} & \frac{6D\Delta t^2}{\Delta x^4} & \frac{-4D\Delta t^2}{\Delta x^4} \\ 0 & \dots & 0 & 0 & \frac{D\Delta t^2}{\Delta x^4} & \frac{-4D\Delta t^2}{\Delta x^4} & \frac{7D\Delta t^2}{\Delta x^4} \end{pmatrix}$$

$$C = \begin{pmatrix} -2+\Delta t^2\alpha_1 & 0 & 0 & 0 & 0 \\ 0 & -2+\Delta t^2\alpha_2 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 0 \\ 0 & \dots & 0 & -2+\Delta t^2\alpha_{N-1} & 0 \\ 0 & \dots & 0 & 0 & -2+\Delta t^2\alpha_N \end{pmatrix}$$

Remark: The formulation Arbitrary Lagrangian-Eulerian (ALE) was used, by considering a dynamic grid [4].

4 Numerical results

The boundary conditions imposed to the pressure [7]

$$p_{in}(x,y,t) = \begin{cases} 10000(1-\cos(\frac{\pi t}{0.0025}))\, \forall (x,y) \in \Sigma_1, 0 \le t \le 0.005 \\ 0, \forall (x,y) \in \Sigma_1, 0.005 \le t \le T \end{cases}$$

We take parameters for fluid and structure in [8, 9].

Table 1 Parameters of the strcuture

Paramerters	Values
E	$0.75 \times 10^6 g/cm^2 \cdot s^2$
h	0.1 cm
ν	0.3
ρ^s	$1.1\frac{g}{cm^3}$

Table 2 Parameters of the fluid

Paramerters	Values
μ	$0.035\frac{g}{cm.s}$
L	3 cm
H	0.5 cm
g	$9.81\ \frac{cm}{s^2}$
$f^F = (f_1^F, f_2^F)$	$(0, 0)$
Δt	$1ms$

Description of the computational method

step 1: We compute the velocity and the pressure of the fluid in the reference domain Ω_F^0.
step 2: From $i = 1$ to N (N is the number of iteration).

- compute the displacement u of the structure.
- Determine the deformed domain Ω_F^u.
- compute the velocity and the pressure in Ω_F^u.

Freefem ++ [3] is used for the numerical tests. Figures following display the structure displacement, the pressure.

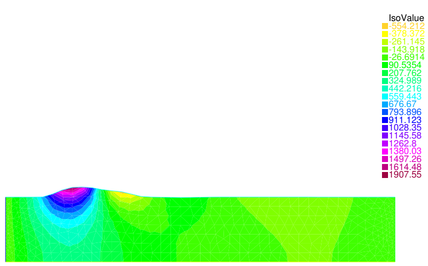

Figure 1: Wave pressure at t = 15ms.

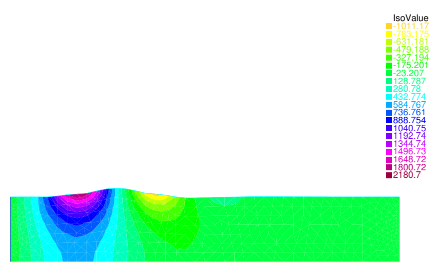

Figure 2: Wave pressure at t = 20ms.

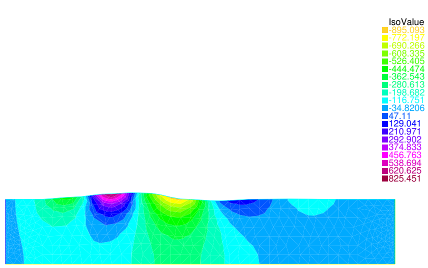

Figure 3: Wave pressure at t = 25ms.

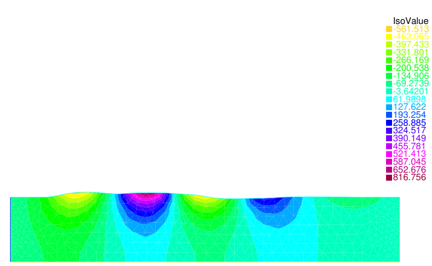

Figure 4: Wave pressure at t = 30ms.

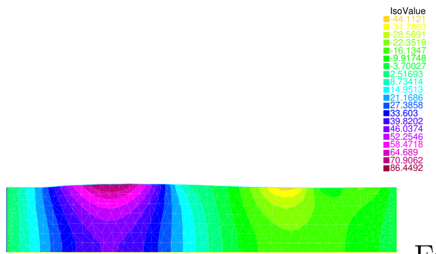

Figure 5: Wave pressure at t = 90ms.

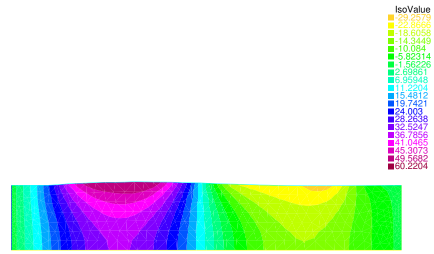

Figure 6: Wave pressure at t = 100ms.

The above figures display the modification step by step of the wave pressure and the structure displacement at t = 15, 20, 25, 30, 90, 100ms.

5 Conclusion

In this paper, we have presented a method to solve fluid structure interaction problem an unsteady case. This paper also comes to improve the work developed in the former paper [2]. On the one hand, we use finite diffence method and leap-flog scheme to compute for each time the displacement of the structure and the other hand we compute the velocity and the pressure of the fluid for each time in the deformed domain. Morever, we assume that for a small displacement and a small step time our computational method is stable. The numerical results are suitable compared to [1]. In our future works we plan to use the same teckniques to solve the problem of benchmark fluid structure interaction.

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