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Homotopy Perturbation Sumudu Transform Method for Nonlinear Equations

Jagdev Singh

Department of Mathematics, Jagan Nath University,
Village-Rampura, Tehsil-Chaksu, Jaipur-303901, Rajasthan, India
jagdevsinghrathore@gmail.com

Devendra Kumar

Department of Mathematics, JaganNath Gupta Institute of Engineering
and Technology, Jaipur-302022, Rajasthan, India
devendra.maths@ gmail.com

Sushila

Department of Physics, Jagan Nath University,
Village-Rampura, Tehsil-Chaksu, Jaipur-303901, Rajasthan, India
sushila.jag@gmail.com

Abstract

In this paper, we propose a combined form of the sumudu transform method with the homotopy perturbation method to solve nonlinear equations. This method is called the homotopy perturbation sumudu transform method (HPSTM). The nonlinear terms can be easily handled by the use of He's polynomials. The proposed scheme finds the solution without any discretization or restrictive assumptions and avoids the round-off errors. The fact that the proposed technique solves nonlinear problems without using Adomian's polynomials can be considered as a clear advantage of this algorithm over the decomposition method. The results reveal that the proposed method is very efficient, simple and can be applied to other nonlinear problems.

Keywords: Sumudu transform, Homotopy perturbation method, Homotopy perturbation sumudu transform method, Nonlinear advection equations, He's Polynomials

1 Introduction

In the last several years with the rapid development of nonlinear science, there has appeared ever-increasing interest of scientists and engineers in the analytical asymptotic techniques for nonlinear problems such as solid state physics, plasma physics, fluid mechanics and applied sciences. In many different fields of science and engineering, it is important to obtain exact or numerical solution of the nonlinear partial differential equations. Searching of exact and numerical solution of nonlinear equations in science and engineering is still quite problematic that's need new methods for finding the exact and approximate solutions. Most of new nonlinear equations do not have a precise analytic solution; so, numerical methods have largely been used to handle these equations. There are also analytic techniques for nonlinear equations. Some of the classic analytic methods are Lyapunov's artificial small parameter method [1], δ-expansion method [2], perturbation techniques [3-5] and Hirota bilinear method [6, 7]. In recent years, many research workers have paid attention to study the solutions of nonlinear partial differential equations by using various methods. Among these are the Adomian decomposition method (ADM) [8], He's semi-inverse method [9], the tanh method, the homotopy perturbation method (HPM), the sinh–cosh method, the differential transform method and the variational iteration method (VIM) [10-17]. Several techniques including the Adomian decomposition method, the variational iteration method, the weighted finite difference techniques and the Laplace decomposition method have been used to handle advection equations [18-24]. Most of these methods have their inbuilt deficiencies like the calculation of Adomian's polynomials, the Lagrange multiplier, divergent results and huge computational work. He [25-33] developed the homotopy perturbation method (HPM) by merging the standard homotopy and perturbation for solving various physical problems. It is worth mentioning that the HPM is applied without any discretization, restrictive assumption or transformation and is free from round off errors. The Laplace transform is totally incapable of handling nonlinear equations because of the difficulties that are caused by the nonlinear terms. Various ways have been proposed recently to deal with these nonlinearities such as the Adomian decomposition method [39] and the Laplace decomposition algorithm [40-44]. Furthermore, the homotopy perturbation method is also combined with the well-known Laplace transformation method [35, 45] and the variational iteration method [46] to produce a highly effective technique for handling many nonlinear problems.

In the present paper, we propose a new method called homotopy perturbation sumudu transform method (HPSTM) for solving the nonlinear equations. It is worth mentioning that the proposed method is an elegant combination of the sumudu transformation, the homotopy perturbation method and He's polynomials and is mainly due to Ghorbani [36, 37]. The use of He's polynomials in the nonlinear term was first introduced by Ghorbani [36, 37]. The proposed algorithm provides the solution in a rapid convergent series which may lead to the solution in a closed form. The advantage of this method is its capability of combining two powerful methods for obtaining exact solutions for nonlinear equations. This article considers the effectiveness of the homotopy perturbation sumudu transform method (HPSTM) in solving nonlinear advection equations, both homogeneous and non-homogeneous.

2 Sumudu transform

In early 90's, Watugala [34] introduced a new integral transform, named the sumudu transform and applied it to the solution of ordinary differential equation in control engineering problems. The Sumudu transform is defined over the set of functions

$$\mathrm{A} = \{\mathrm{f(t)} \mid \exists\, \mathrm{M}, \tau_1, \tau_2 > 0,\ |\,\mathrm{f\,(t)}\,| < \mathrm{M\,e}^{|\mathrm{t}|/\tau_{\mathrm{j}}},\ \text{if t} \in (-1)^{\mathrm{j}} \times [0,\infty)\}$$

by the following formula

$$\bar{\mathrm{f}}(\mathrm{u}) = \mathrm{S\,[f(t)]} = \int_0^{\infty} \mathrm{f\,(ut)\,e^{-t}\,dt},\ \mathrm{u} \in (-\tau_1, \tau_2). \tag{1}$$

For further detail and properties of this transform, see [47-49].

3 Homotopy perturbation sumudu transform method (HPSTM)

To illustrate the basic idea of this method, we consider a general nonlinear non-homogenous partial differential equation with the initial conditions of the form:

$$D\,U(x,t) + RU(x,t) + N\,U(x,t) = g(x,t), \tag{2}$$
$$U(x,0) = h(x), \quad U_t(x,0) = f(x),$$

where D is the second order linear differential operator $D = \partial^2/\partial t^2$, R is the linear differential operator of less order than D, N represents the general nonlinear differential operator and $g(x,t)$ is the source term.
Taking the sumudu transform on both sides of eq. (2), we get

$$S\,[D\,U(x,t)] + S\,[R\,U(x,t)] + S\,[N\,U(x,t)] = S\,[g(x,t)]. \tag{3}$$

Using the differentiation property of the sumudu transform and above initial conditions, we have

$$S\,[U(x,t)] = u^2 S\,[g(x,t)] + h(x) + uf(x) - u^2 S\,[R\,U(x,t) + N\,U(x,t)]\,. \tag{4}$$

Now, applying the inverse sumudu transform on both sides of eq. (4), we get

$$U(x,t) = G(x,t) - S^{-1}\left[u^2\, S\left[R\, U(x,t) + N\, U(x,t)\right]\right], \tag{5}$$

where $G(x,t)$ represents the term arising from the source term and the prescribed initial conditions. Now, we apply the homotopy perturbation method

$$U(x,t) = \sum_{n=0}^{\infty} p^n U_n(x,t) \tag{6}$$

and the nonlinear term can be decomposed as

$$N\, U(x,t) = \sum_{n=0}^{\infty} p^n H_n(U), \tag{7}$$

for some He's polynomials $H_n(U)$(see [37, 38]) that are given by

$$H_n(U_0, U_1, ..., U_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i U_i\right)\right]_{p=0}, n = 0, 1, 2, 3, ... \tag{8}$$

Substituting eqs. (6) and (7) in eq. (5), we get

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = G(x,t) - p\left(S^{-1}\left[u^2\, S\left[R\sum_{n=0}^{\infty} p^n U_n(x,t) + \sum_{n=0}^{\infty} p^n H_n(U)\right]\right]\right), \tag{9}$$

which is the coupling of the sumudu transform and the homotopy perturbation method using He's polynomials.
Comparing the coefficient of like powers of p, the following approximations are obtained.

$$\begin{aligned} p^0 &: \ U_0(x,t) = G(x,t),\\ p^1 &: \ U_1(x,t) = -S^{-1}\left[u^2 S\left[R\, U_0(x,t) + H_0(U)\right]\right],\\ p^2 &: \ U_2(x,t) = -S^{-1}\left[u^2 S\left[R\, U_1(x,t) + H_1(U)\right]\right],\\ p^3 &: \ U_3(x,t) = -S^{-1}\left[u^2 S\left[R\, U_2(x,t) + H_2(U)\right]\right],\\ &\ \ \vdots \end{aligned} \tag{10}$$

4 Applications

In order to elucidate the solution procedure of the homotopy perturbation sumudu transform method (HPSTM), we first consider the nonlinear advection equations.

Example 4.1. Consider the following homogenous advection problem [19, 23]:

$$U_t + UU_x = 0, \tag{11}$$
$$U(x,\,0) = -x.$$

Taking the sumudu transform on both sides of eq. (11) subject to the initial condition, we have

$$S[U(x,t)] = -x - u\,S[UU_x]. \tag{12}$$

The inverse of sumudu transform implies that

$$U(x,t) = -x - S^{-1}\left[u\,S[UU_x]\right]. \tag{13}$$

Now, applying the homotopy perturbation method, we get

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = -x - p\left(S^{-1}\left[u\,S\left[\sum_{n=0}^{\infty} p^n H_n(U)\right]\right]\right), \tag{14}$$

where $H_n(U)$ are He's polynomials [37, 38] that represents the nonlinear terms. The first few components of He's polynomials, are given by

$$\begin{aligned} H_0(U) &= U_0U_{0x}, \\ H_1(U) &= U_0U_{1x} + U_1U_{0x}, \\ H_2(U) &= U_0U_{2x} + U_1U_{1x} + +U_2U_{0x}, \\ &\vdots \end{aligned} \tag{15}$$

Comparing the coefficients of like powers of p, we hav

$$\begin{aligned} p^0 &: U_0(x,t) = -x, \\ p^1 &: U_1(x,t) = -S^{-1}\left[u\,S[H_0(U)]\right] = -x\,t, \\ p^2 &: U_2(x,t) = -S^{-1}\left[u\,S[H_1(U)]\right] = -x\,t^2. \end{aligned} \tag{16}$$

Proceeding in a similar manner, we have

$$\begin{aligned} p^3 &: U_3(x,t) = -x\,t^3, \\ p^4 &: U_4(x,t) = -x\,t^4, \\ &\vdots \end{aligned} \tag{17}$$

Therefore the solution $U(x,t)$ is given by

$$U(x,t) = -x(1 + t + t^2 + t^3 + t^4 + \cdots), \tag{18}$$

in series form, and

$$U(x,t) = \frac{x}{t-1}, \tag{19}$$

in closed form.

Example 4.2. Now, consider the following nonhomogenous advection problem [19, 23]:

$$U_t + UU_x = 2t + x + t^3 + xt^2, \tag{20}$$
$$U(x,\, 0) = 0.$$

Taking the sumudu transform on both sides of eq. (20) subject to the initial condition, we have

$$S[U(x,t)] = 2u^2 + xu + 6u^4 + 2xu^3 - u\, S[UU_x]. \tag{21}$$

The inverse of sumudu transform implies that

$$U(x,t) = t^2 + xt + \frac{t^4}{4} + \frac{xt^3}{3} - S^{-1}\left[u\, S[UU_x]\right]. \tag{22}$$

Now, applying the homotopy perturbation method, we get

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = t^2 + xt + \frac{t^4}{4} + \frac{xt^3}{3} - p\left(S^{-1}\left[u\, S\left[\sum_{n=0}^{\infty} p^n H_n(U)\right]\right]\right), \tag{23}$$

where $H_n(U)$ are He's polynomials [37, 38] that represents the nonlinear terms. The first few components of He's polynomials, are given by

$$\begin{aligned} H_0(U) &= U_0 U_{0x}, \\ H_1(U) &= U_0 U_{1x} + U_1 U_{0x}, \\ H_2(U) &= U_0 U_{2x} + U_1 U_{1x} + +U_2 U_{0x}, \\ &\vdots \end{aligned} \tag{24}$$

Comparing the coefficients of like powers of p, we have

$$\begin{aligned} p^0 &: U_0(x,t) = t^2 + xt + \frac{t^4}{4} + \frac{xt^3}{3}, \\ p^1 &: U_1(x,t) = -\frac{1}{4}t^4 - \frac{1}{3}xt^3 - \frac{2}{15}xt^5 - \frac{7}{72}t^6 - \frac{1}{63}xt^7 - \frac{1}{98}t^8, \\ p^2 &: U_2(x,t) = \frac{5}{8064}t^{12} + \frac{2}{2079}xt^{11} + \frac{2783}{302400}t^{10} + \frac{38}{2835}xt^9 + \frac{143}{2880}t^8 + \\ &\qquad \frac{22}{315}xt^7 + \frac{7}{12}t^6 + \frac{2}{15}xt^{5,} \end{aligned} \tag{25}$$

$$\vdots$$

It is important to recall here that the noise terms appear between the components $U_0(x,t)$ *and* $U_1(x,t)$, where the noise terms are those pairs of terms that are identical but carrying opposite signs. More precisely, the noise terms $\pm\frac{1}{4}t^4 \pm \frac{1}{3}xt^3$ between the components $U_0(x,t)$ *and* $U_1(x,t)$ can be cancelled and the remaining terms of $U_0(x,t)$ still satisfy the equation. Therefore, the exact solution is given by

$$U(x,t) = t^2 + xt. \tag{26}$$

5 Conclusions

In the present paper, we heve proposed the homotopy perturbation sumudu transform method (HPSTM) for solving nonlinear problems. In previous papers [19, 23, 39-44] many authors have already used Adomian polynomials to decompose the nonlinear terms in equations. The solution procedure is simple, but the calculation of Adomian polynomials is complex. To overcome this shortcoming, we proposed a new approach using He's polynomials [37, 38]. It is worth mentioning that the method is capable of reducing the volume of the computational work as compared to the classical methods while still maintaining the high accuracy of the numerical result; the size reduction amounts to an improvement of the performance of the approach. The fact that the HPSTM solves nonlinear problems without using Adomian's polynomials is a clear advantage of this technique over the decomposition method. In conclusion, the HPSTM may be considered as a nice refinement in existing numerical techniques and might find the wide applications.

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