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Birkhoff Centre of an Almost Distributive Lattice

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Abstract

In this paper, the concept of the Birkhoff centre $B(L)$ of an Almost Distributive Lattice L with maximal elements is introduced and proved that $B(L)$ is a relatively complemented ADL. Mainly it is proved that the elements of $B(L)$ are in one-to-one correspondence with the complemented ideals of L and also with the factor-congruences on L and hence with the direct decompositions of L.

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1 Introduction

The concept of an Almost Distributive Lattice was introduced by U.M.Swamy and G.C.Rao [4] in 1980. It is an algebraic structure which satisfies almost all the properties of a distributive lattice with the smallest element except the commutativity of the operations $\vee$ and $\wedge$ and the right distributivity of $\vee$ over $\wedge$. It is well known that the Birkhoff centre of a bounded partially ordered set P is a Boolean algebra in which the operations are l.u.b. and g.l.b. in P [1]. In [4], U.M.Swamy, G.C.Rao, R.V.G.Ravi Kumar and Ch. Pragathi have extended the above concept for a general partially ordered set P and proved that $B(P)$ is a relatively complemented distributive lattice in which the operations are l.u.b. and g.l.b, in P (provided $B(P)$ is non-empty).

Also, they have observed that, for a lattice, Birkhoff centres as a lattice and as a partially ordered set coincide. In this paper, we introduce the concept of the Birkhoff centre $B(L)$ of an Almost Distributive Lattice L with maximal elements and prove that $B(L)$ is a relatively complemented almost distributive lattice. Mainly we obtain a one-to-one correspondences between the Birkhoff centre of L and the set of complemented ideals of L and between the Birkhoff centre of L and the set of factor-congruences on L.

2 Preliminaries

In this section we recollect some preliminary concepts and results on Almost Distributive Lattices from [2].

Definition 2.1. *[2] An algebra $(L, \vee, \wedge, 0)$ of type (2, 2, 0) is said to be an Almost Distributive Lattice (ADL) if it satisfies the following conditions.*

(1) $a \vee 0 = a$
(2) $0 \wedge a = 0$
(3) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$
(4) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
(5) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
(6) $(a \vee b) \wedge b = b$

for all $a, b, c \in L$. The element 0 is called, as usual, the zero element of L.

Example 2.2. *[2] Let X be a non-empty set. Fix $x_0 \in X$. For any $x, y \in X$, define*

$$x \wedge y = \begin{cases} x_0 & \text{if } x = x_0 \\ y & \text{if } x \neq x_0 \end{cases} \qquad \text{and} \qquad x \vee y = \begin{cases} y & \text{if } x = x_0 \\ x & \text{if } x \neq x_0 \end{cases}$$

Then $(X, \vee, \wedge, x_0)$ is an ADL with x_0 as its zero element. This ADL is called a discrete ADL.

Example 2.3. *[2] Let $(R, +, \cdot, 0, 1)$ be a commutative regular ring with identity. For any $a \in R$, let a_0 be an idempotent element such that $a_0R = aR$. For any $x, y \in R$, define $x \wedge y = x_0y$ and $x \vee y = x + y + x_0y$. Then $(R, \vee, \wedge, 0)$ is an ADL.*

Example 2.4. *Every distributive lattice with zero is an ADL.*

For any a, b in an ADL L, we say that a is less than or equal to b and write $a \leq b$, if $a \wedge b = a$. Then $\leq$ is a partial ordering on L.

Throughout this paper, unless otherwise stated, L denotes an ADL $(L, \vee, \wedge, 0)$.

Lemma 2.5. *[2] For any $a, b \in L$, we have*
(1) $a \wedge 0 = 0$ *and* $0 \vee a = a$
(2) $a \wedge a = a = a \vee a$
(3) $(a \wedge b) \vee b = b, a \vee (b \wedge a) = a$ *and* $a \wedge (a \vee b) = a$
(4) $a \wedge b = b \iff a \vee b = a$
(5) $a \wedge b = a \iff a \vee b = b$
(6) $a \wedge b \leq b$ *and* $a \leq a \vee b$
(7) $a \wedge b = b \wedge a$ *whenever* $a \leq b$
(8) $a \vee (b \vee a) = a \vee b$.

Theorem 2.6. *[2] For any $a, b \in L$, the following are equivalent to each other.*
(1) $(a \wedge b) \vee a = a$
(2) $a \wedge (b \vee a) = a$
(3) $(b \wedge a) \vee b = b$
(4) $b \wedge (a \vee b) = b$
(5) $a \wedge b = b \wedge a$
(6) $a \vee b = b \vee a$
(7) The supremum of a and b exists and equals to $a \vee b$
(8) There exists $x \in L$ such that $a \leq x$ and $b \leq x$
(9) The infimum of a and b exists and equals to $a \wedge b$.

Theorem 2.7. *[2] For any $a, b, c \in L$, we have*
(1) $(a \vee b) \wedge c = (b \vee a) \wedge c$
(2) $\wedge$ *is associative in* L
(3) $a \wedge b \wedge c = b \wedge a \wedge c$.

From the above theorem, it follows that, for any $x \in L$, the set $\{a \wedge x \mid a \in L\}$ forms a bounded distributive lattice and, in particular, we have

$$((a \wedge b) \vee c) \wedge x = ((a \vee c) \wedge (b \vee c)) \wedge x$$

for all $a, b, c, x \in L$. An element $m \in L$ is said to be maximal if $m \leq x$ implies $m = x$. It can be easily observed that m is maximal if and only if $m \wedge x = x$ for all $x \in L$.

Definition 2.8. *[2] A non-empty subset I of L is said to be an ideal of L if it satisfies the following;*
(i) $a, b \in I \Rightarrow a \vee b \in I$
(ii) $a \in I, x \in L \Rightarrow a \wedge x \in I$.

If I is an ideal of L, then $x \wedge a \in I$ for any $a \in I$ and $x \in L$; For, $x \wedge a = x \wedge a \wedge a = a \wedge x \wedge a \in I$. Therefore in this case, any right ideal in the usual sense is a left ideal two and hence a two sided ideal in the usual sense. However a left ideal may not be a right ideal. For consider the following example.

Example 2.9. *Let D be a discrete ADL. For any $0 \neq x \in D$, the set $\{0, x\}$ is a left ideal but not a right ideal of D.*

Definition 2.10. *[2] A non-empty subset F of L is said to be a filter of L, if it satisfies the following;*
(i) $a, b \in F \Rightarrow a \wedge b \in F$
(ii) $a \in F, x \in L \Rightarrow x \vee a \in F$.

If F is a filter of L, then $a \vee x \in F$ for any $a \in F$ and $x \in L$; For, $a \vee x = (a \vee x) \wedge (a \vee x) = (x \vee a) \wedge (a \vee x) = (x \wedge (a \vee x)) \vee (a \wedge (a \vee x)) = (x \wedge (a \vee x)) \vee a \in F$. Therefore in any ADL, every left filter in the usual sense is a right filter and hence a two sided filter in the usual sense. However a right filter may not be a left filter. For, consider the following example.

Example 2.11. *Let D be a discrete ADL. For any $0 \neq x \in D$, the set $\{x\}$ is a right filter but not a left filter of D.*

It is known that, for any $x, y \in L$ with $x \leq y$, the interval $[x, y]$ is a bounded distributive lattice. Now, an ADL L is said to be relatively complemented if, for any $x, y \in L$ with $x \leq y$, the interval $[x, y]$ is a complemented distributive lattice.

Theorem 2.12. *[2] The following are equivalent for any ADL L;*
(1) L is relatively complemented
(2) Given $x, y \in L$, there exists $a \in L$ such that $x \wedge a = 0$ and $x \vee a = x \vee y$
(3) For any $x \in L$, the interval $[0, x]$ is complemented.

3 The Birkhoff Centre

In this section we define the Birkhoff centre of an Almost Distributive Lattice L with maximal elements and prove that the Birkhoff centre of L is a relatively complemented ADL. We obtain a one-to-one correspondences between the set $B(L)$, of complemented ideals of L and the set of factor-congruences on L. Throughout this paper we consider only ADLs which contain atleast one maximal element.

Definition 3.1. *Given an ADL L, define*
$B(L) := \{a \in L \mid$ *there exists $b \in L$ such that $a \wedge b = 0$ and $a \vee b$ is maximal*$\}$.

If $a \wedge b = 0$ and $a \vee b$ is maximal, then $b \wedge a = 0$ and $b \vee a$ is maximal; in this case, a and b are called complements to each other. Note that complement of an element need not be unique; for example, in a discrete ADL, every non-zero element is a complement of 0. If L_1 and L_2 are ADLs with maximal elements, then it can be easily verified that $L_1 \times L_2$ is so; Infact (m_1, m_2) is a maximal element in $L_1 \times L_2$ if and only if m_1 and m_2 are maximal elements in L_1 and L_2 respectively. In the following we extend the result on a bounded distributive lattice L corresponding to decompositions of L into products of two bounded distributive lattices.

Theorem 3.2. *For any $a \in L$, $a \in B(L)$ if and only if there exist two ADLs L_1 and L_2 with maximal elements and an isomorphism $f : L \to L_1 \times L_2$ such that $f(a) = (m_1, 0)$, where m_1 is a maximal element in L_1.*

Proof. Suppose that $a \in B(L)$. Then there exists $b \in L$ such that $a \wedge b = 0$ and $a \vee b$ is maximal.

Put $L_1 = (a) = \{a \wedge x \mid x \in L\}$ and $L_2 = (b) = \{b \wedge x \mid x \in L\}$
Then L_1 and L_2 are ADLs (subADLs of L) and a and b are maximal elements of L_1 and L_2 respectively. Define $f : L \to L_1 \times L_2$ by

$$f(x) = (a \wedge x, b \wedge x) \text{ for all } x \in L.$$

Then f is an isomorphism from L onto $L_1 \times L_2$ such that $f(a) = (a, 0)$, and a is a maximal element of L_1. Conversely suppose that there exist two ADLs L_1 and L_2 with maximal elements and an isomorphism $f : L \to L_1 \times L_2$ such that $f(a) = (m_1, 0)$, where m_1 is a maximal element of L_1. Choose a maximal element m_2 in L_2. Then there exists $b \in L$ such that $f(b) = (0, m_2)$. Now, $f(a \wedge b) = f(a) \wedge f(b) = (m_1, 0) \wedge (0, m_2) = (0, 0) = f(0)$ and $f(a \vee b) = f(a) \vee f(b) = (m_1, 0) \vee (0, m_2) = (m_1, m_2)$ which is maximal in $L_1 \times L_2$. Therefore $a \wedge b = 0$ and $a \vee b$ is maximal. Thus $a \in B(L)$. □

In the following we observe that the Birkhoff centre of L is a relatively complemented subADL of L.

Theorem 3.3. *$B(L)$ is a relatively complemented ADL under the operations induced by those of L.*

Proof. Clearly $0 \in B(L)$ and hence $B(L)$ is a non-empty subset of L. Let a_1 and $a_2 \in B(L)$. Let b_1 and $b_2 \in L$ be complements of a_1 and a_2 respectively; that is, $a_1 \wedge b_1 = 0 = a_2 \wedge b_2$ and $a_1 \vee b_1, a_2 \vee b_2$ are maximal.

$$\begin{aligned}(a_1 \wedge a_2) \wedge (b_1 \vee b_2) &= (a_1 \wedge a_2 \wedge b_1) \vee (a_1 \wedge a_2 \wedge b_2)\\ &= (a_1 \wedge b_1 \wedge a_2 \wedge b_1) \vee (a_1 \wedge (a_2 \wedge b_2))\\ &= 0 \qquad (\because a_1 \wedge b_1 = 0 = a_2 \wedge b_2)\end{aligned}$$

and, for all $x \in L$,

$$\begin{aligned}((a_1 \wedge a_2) \vee (b_1 \vee b_2)) \wedge x &= [(a_1 \vee b_1 \vee b_2) \wedge (a_2 \vee b_1 \vee b_2)] \wedge x\\ &= x \qquad \text{(since } a_1 \vee b_1 \text{ and } a_2 \vee b_2 \text{ are maximal)}\end{aligned}$$

and hence $(a_1 \wedge a_2) \vee (b_1 \vee b_2)$ is maximal. Therefore $b_1 \vee b_2$ is a complement of $a_1 \wedge a_2$ and hence $a_1 \wedge a_2 \in B(L)$. From this, it follows that $b_1 \wedge b_2$ is a complement of $a_1 \vee a_2$ and hence $a_1 \vee a_2 \in B(L)$. Therefore $B(L)$ is an ADL under the operations induced by those of L. Let $a, b \in B(L)$. Then there exist $c, d \in L$ such that $a \wedge c = 0 = b \wedge d$ and $a \vee c$ and $b \vee d$ are maximal. Put $x = c \wedge b$ and $y = a \vee d$. Then

$$\begin{aligned} x \wedge y &= c \wedge b \wedge (a \vee d) \\ &= (c \wedge b \wedge a) \vee (c \wedge b \wedge d) \\ &= 0 \qquad \text{(since } a \wedge c = 0 = b \wedge d) \end{aligned}$$

and, for any $t \in L$,

$$\begin{aligned} (x \vee y) \wedge t &= ((c \wedge b) \vee (a \vee d)) \wedge t \\ &= (c \vee a \vee d) \wedge (b \vee a \vee d) \wedge t \\ &= (a \vee c) \wedge (b \vee d) \wedge t \\ &= t \qquad \text{(since } a \vee c \text{ and } b \vee d \text{ are maximal).} \end{aligned}$$

Therefore $x \in B(L)$. Now, $a \wedge x = a \wedge c \wedge b = 0$ and $a \vee x = a \vee (c \wedge (a \vee b)) = (a \vee c) \wedge (a \vee b) = a \vee b$ (since $a \vee c$ is maximal). Therefore $B(L)$ is a relatively complemented ADL. □

In the following we observe that the Birkhoff centre of a relatively complemented ADL with maximal elements is equal to itself.

Theorem 3.4. *L is relatively complemented if and only if $B(L) = L$.*

Proof. Suppose L is relatively complemented. Let $x \in L$ such that x is not maximal. Then there exists a maximal element m of L such that $x \leq m$. (for example, if n is maximal in L, then $x \vee n$ is also maximal and $x \leq x \vee n$). Since L is relatively complemented, there exists $y \in L$ such that $x \wedge y = 0$ and $x \vee y = m$. Therefore $x \in B(L)$ and hence $B(L) = L$. The converse follows from theorem 3.3. □

The following is a straightforward verification.

Theorem 3.5. *If L_1 and L_2 are ADLs, then $B(L_1 \times L_1) = B(L_1) \times B(L_2)$.*

Let L be an ADL. The relation $\eta := \{(a, b) \in L \times L \mid a \wedge b = b \text{ and } b \wedge a = a\}$ is a congruence relation on L and is the smallest such that L/η is a lattice. We have the following;

Theorem 3.6. *$B(L/\eta)$ is isomorphic to $B(L)/\eta_{|B(L) \times B(L)}$.*

Proof. Let $a/\eta \in B(L/\eta)$. Then there exists $b \in L$ such that $a/\eta \wedge b/\eta = 0/\eta$ and $a/\eta \vee b/\eta$ is maximal in L/η. Therefore $a \wedge b = 0$ and $(a \vee b)/\eta$ is maximal in L/η. Now, for any $x \in L$, we have $((a \vee b) \wedge x)/\eta = (a \vee b)/\eta \wedge x/\eta = x/\eta$. Then $((a \vee b) \wedge x, x) \in \eta$ and hence $(a \vee b) \wedge x = x$. Therefore $a \vee b$ is maximal in L. Hence $a \in B(L)$. Consider the map $f : B(L) \to B(L/\eta)$ defined by $f(a) = a/\eta$ for any $a \in B(L)$. Then f is an epimorphism and $Kerf = \eta \cap (B(L) \times B(L))$. Hence by the fundamental theorem of homomorphisms, $B(L)/\eta_{|B(L) \times B(L)} \cong B(L/\eta)$. □

It is known that an ideal I of an ADL L is complemented if and only if $I = (a)$ for some $a \in L$ [2]. Infact, we have the following.

Theorem 3.7. *An ideal I of L is complemented if and only if $I = (a)$ for some $a \in B(L)$.*

Proof. Let I be an ideal of L. Suppose that I is complemented. Then there exists an ideal I' of L such that $I \cap I' = (0)$ and $I \vee I' = L$. Choose a maximal element m of L. Then $m = a \vee b$ for some $a \in I$ and $b \in I'$. Since $I \cap I' = (0)$, $a \wedge b = 0$ and we have that $a \vee b$ is maximal. Therefore $a \in B(L)$. Now, since $a \in L$, we have that $(a) \subseteq I$. Also,

$$\begin{aligned} x \in I \ &\Rightarrow\ b \wedge x \in I \cap I' = (0) \\ &\Rightarrow\ b \wedge x = 0 \\ &\Rightarrow\ x = m \wedge x = (a \vee b) \wedge x = a \wedge x \\ &\Rightarrow\ x \in (a). \end{aligned}$$

Therefore $(a) = I$. Similarly $(b) = I'$. Conversely, suppose that $I = (a)$ for some $a \in B(L)$. Then there exists $b \in L$ such that $a \wedge b = 0$ and $a \vee b$ is maximal. Now, $(a) \cap (b) = (0)$ and $(a) \vee (b) = (a \vee b) = L$. Therefore I is complemented. □

Given any filter F of an ADL L, define

$$\phi_F := \{(a, b) \in L \times L \mid x \wedge a = x \wedge b \text{ for some } x \in F\}.$$

Then ϕ_F is a congruence on L. We write $\phi_x := \{(a, b) \in L \times L \mid x \wedge a = x \wedge b\}$, for any $x \in L$.

The following is a routine verification.

Theorem 3.8. *We have the following;*
(i) For any filters F, G of L, $\phi_F \cap \phi_G = \phi_{F \cap G}$ and $\phi_F \ o\ \phi_G = \phi_{F \vee G}$
(ii) For any $x \in L$, $\phi_{[x)} = \phi_x$, where $[x) := \{y \vee x \mid y \in L\}$
(iii) For any $x \in L$, $\phi_x = \Delta$ if and only if x is maximal
(iv) For any $x \in L$, $\phi_x = L \times L$ if and only if $x = 0$.

A congruence θ on an ADL L is said to be a factor-congruence on L if there exists a congruence ϕ on L such that $\theta \cap \phi = \Delta$ and $\theta \ o\ \phi = L \times L$; or, equivalently, $x \mapsto (\theta(x), \phi(x))$ is an isomorphism of L onto $L/\theta \times L/\phi$. In other words, factor congruences correspond to direct decompositions of L. The following gives a correspondence between factor congruences on L and elements of $B(L)$ (see Theorem 3.2).

Theorem 3.9. *A congruence θ on L is a factor-congruence if and only if $\theta = \phi_a$ for some $a \in B(L)$.*

Proof. Suppose that θ is a factor-congruence on L. Then there exists a congruence ϕ on L such that $\theta \cap \phi = \Delta$ and $\theta \ o\ \phi = L \times L$. Choose a maximal element m of L. Then $(m, 0) \in \theta \ o\ \phi$ and hence there exists $b \in L$ such that $(m, b) \in \phi$

and $(b, 0) \in \theta$. Also, $(0, m) \in \theta \; o \; \phi$ and hence there exists $a \in L$ such that $(0, a) \in \phi$ and $(a, m) \in \theta$. Now, $(0, b \wedge a) \in \theta \cap \phi = \Delta$. Therefore $a \wedge b = 0$. Since $(m, b) \in \phi$, $(m, b \vee a) \in \phi$. Since $(b \vee a, a), (a, m) \in \theta$, $(m, b \vee a) \in \theta$. Therefore $(m, b \vee a) \in \theta \cap \phi$ and hence $b \vee a = m$, which is maximal. Thus $a \in B(L)$. Now, we will show that $\phi_a = \theta$. Let $x, y \in L$. If $(x, y) \in \phi_a$, then $a \wedge x = a \wedge y$. Since $(a, m) \in \theta$, we get that $(a \wedge x, x), (a \wedge y, y) \in \theta$. Since $a \wedge x = a \wedge y$, $(x, y) \in \theta$. Therefore $\phi_a \subseteq \theta$. If $(x, y) \in \theta$, then $(a \wedge x, a \wedge y) \in \theta$. Since $(0, a) \in \phi$, $(0, a \wedge x), (0, a \wedge y) \in \phi$. Therefore $(a \wedge x, a \wedge y) \in \phi \cap \theta = \Delta$ and hence $a \wedge x = a \wedge y$. Hence $(x, y) \in \phi_a$. Thus $\phi_a = \theta$ and $a \in B(L)$. Conversely, suppose that $\theta = \phi_a$ for some $a \in B(L)$. Then there exists $b \in L$ such that $a \wedge b = 0$ and $a \vee b$ is maximal. Now,

$\phi_a \cap \phi_b = \phi_{[a)} \cap \phi_{[b)} = \phi_{[a) \cap [b)} = \phi_{[a \vee b)} = \phi_{a \vee b} = \Delta$ (since $a \vee b$ is maximal)

$\phi_a \; o \; \phi_b = \phi_{[a)} \; o \; \phi_{[b)} = \phi_{[a) \vee [b)} = \phi_{[a \wedge b)} = \phi_{a \wedge b} = \phi_0 = L \times L$.

Therefore θ is a factor-congruence on L. □

Corollary 3.10. *L is relatively complemented if and only if ϕ_a is a factor congruence for every $a \in L$.*

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