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Controllability of Impulsive Partial Neutral Functional Differential Equation with Infinite Delay

D. N. Chalishajar

Department of Mathematics, 417 Mallory Hall
Virginia Military Institute (VMI)
Lexington, VA 24450, USA
or
343 Department of Mathematics and Statistics
James Madison University
Harrisonburg, VA 22807, USA
dipu17370@yahoo.com

Abstract

In this paper, we examine sufficient condition for controllability of first order impulsive partial neutral functional differential equations. Here we do not assume that the system generates a compact semigroup, so method is applicable to a wide class of impulsive partial neutral functional differential equations in Banach spaces. Also we claim that phase space for infinite delay with impulse, considered by different authors are not correct.

Mathematics Subject Classifications: 34A37, 34G20, 34K20, 93B05, 93C10

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1 Introduction

In this paper, we will prove controllability of following first order impulsive partial neutral functional differential equations of the form:

$$\begin{cases} \frac{d}{dt}[x(t)+g(t,x_t)] = Ax(t)+Bu(t)+f(t,x_t);\ t \in J := [0,b], t \neq t_k; \\ x_0 = \phi \in \mathcal{B}_h; \\ \Delta x(t_k) = I_k(x(t_k));\ k = 1,2,....m; \end{cases} \tag{1}$$

where the linear operator A is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ on a Banach space X. Also the control function $u \in L^2(J,U)$, a Banach space of admissible control functions with U as a Banach space. B is a bounded linear operator from U to X. $x_t : (-\infty, 0] \to X;\ x_t(\theta) = x(t+\theta)$, belongs to some abstract phase space $\mathcal{B}_h$ defined axiomatically in later part of Section 1. $g, f, I_k^j,\ k = 1,2,...,m;\ j = 1,2;$ are appropriate functions; $0 < t_1 < t_2 < ...... < t_m < b$ are fixed numbers and $\Delta x_{t=t_k} = x(t_k^+) - x(t_k^-)$ represents the jump of the function x at $t = t_k$. Since many engineering and real life systems arising from realistic models heavily depend on histories which obey effect of infinite delay on state equation of the system, it is required to study the system of infinite delay with impulse effect extensively.

To study the abstract functional differential equations/inclusions with infinite delay, people usually employ an axiomatic definition of the phase space introduced by Hale and Kato [5], but as defined by Hino, Murakami and Naito [6], these axioms are not correct for the impulsive case. The controllability result for the first order impulsive neutral functional differential inclusions with infinite delay was studied by Bing Liu [8] but with compactness assumption of the semigroup $T(t)_{t\geq 0}$, which leads the system to finite dimension (refer R. Triggiani, [12]). Thus in [8] author claims the result in infinite dimension, is not correct (refer [2]). On the other hand recently Hernandez [7] proved the existence of impulsive partial neutral functional differential equations of first and second order without compactness of semigroup/cosine operators but with the usual axiomatic definition of the phase space introduced by Hale and Kato [5], which is also not correct. Also the examples considered by Hernandez et.al. [7] recover the special case of the abstract results followed by the phase space given by Hale and Kato [5] but not for the impulse case as discussed in Hino, Murakami and Naito [6].

The aim of this paper is to prove controllability result for first order partial neutral functional differential equations with infinite delay and impulse effect within phase

space defined in [6] and without compactness assumption of semigroup/cosine operators. To the best of our knowledge no result is available in this connection.
In this paper we will discuss controllability result for first order partial neutral functional differential equations with impulses and infinite delay, modelled as the initial value problems (1) on a Banach space X of infinite dimension. To describe our problem appropriately we say that a function $w : [\sigma,\tau] \to X$ is a normalized piecewise continuous function on $[\sigma,\tau]$ if w is piecewise continuous function and left continuous on $(\sigma,\tau]$. We denote by $\mathcal{PC}([\sigma,\tau];\ \mathrm{X})$ the space formed by the normalized piecewise continuous functions from $[\sigma,\tau]$ into X. In particular, we introduce the space $\mathcal{PC}$ formed by all functions $w : [0,b] \to X$ such that w is continuous at $t \neq t_k, w(t_k^-) = w(t_k)$ and $w(t_k^+)$ exists, for all $k = 1,2,...,m$. It is clear that $\mathcal{PC}$ endowed with the norm of the uniform convergence is a Banach space.
To consider the impulsive conditions it is convenient to introduce some additional notations. In what follows we put $t_0 = 0, t_{m+1} = b$ for $w \in \mathcal{PC}$ we denote by

$$\widetilde{w_k}(t) = \begin{cases} w(t); & t \in (t_k, t_{k+1}] \\ w(t_k^+); & t = t_k \end{cases}$$

Moreover, for $E \subseteq \mathcal{PC}$ we employ the notation $\widetilde{E_k}, k = 0,1,...,m$; for the sets

$$\widetilde{E_k} = \{\widetilde{w_k} : w \in E\}$$

LEMMA 1.1 *The set $E \subseteq \mathcal{PC}$ is relatively compact if and only if each set $\widetilde{E_k}$ is relatively compact in the space $C([t_k,t_{k+1}];X)$.* ∎

In this work we will employ an axiomatic definition of the phase space $\mathcal{B}_h$ which is similar to that introduced by Hino, Murakami and Naito [6] and appropriate to treat retarded impulsive differential equations. Specifically, $\mathcal{B}_h$ will be a linear space of functions mapping $(-\infty,0]$ into X endowed with a sup-norm $||.||_{\mathcal{B}_h}$.
(A)We present the abstract phase $\mathcal{B}_h$. Assume that $h : (-\infty,0] \to (0,\infty)$ is a continuous function with $l = \int_{-\infty}^0 h(s)ds < \infty$. Define
$\mathcal{B}_h := \{\phi : (-\infty,0] \to X$ for any $r > 0, \phi(\theta)$ is bounded and measurable function on$[-r,0]$ and $\int_{-\infty}^0 h(s)\sup_{s\le\theta\le 0}\|\phi(\theta)\|ds < +\infty\}$.
Here $\mathcal{B}_h$ endowed with the norm

$$\|\phi\|_{\mathcal{B}_h} = \int_{-\infty}^0 h(s)\sup_{s\le 0\le b}\|\phi(\theta)\|ds, \text{for } \phi \in \mathcal{B}_h.$$

Obviously, $(\mathcal{B}_h, \|.\|_{\mathcal{B}_h})$ is a Banach space.
Also we consider the space
$\mathcal{B}_b = \{x : (-\infty, b) \to X$ such that $x_k \in C(J_k, X), x|_J \in \mathcal{PC}$ and $x_0 = \phi \in \mathcal{B}_h\}$,
where x_k is the restriction of x to $J_k = \{t_k, t_{k+1}\}, k = 0, 1, 2, ..., m$; $\|.\|_b$ be a seminorm in $\mathcal{B}_b$ defined by

$$\|x\|_b = \|x_0\|_{\mathcal{B}_h} + sup\{\|x(s)\| : 0 \leq s \leq b\}, x \in \mathcal{B}_b.$$

LEMMA 1.2 *Suppose* $x \in \mathcal{B}_b$, *then* $x_t \in \mathcal{B}_h$ *for* $t \in J$. *Moreover,* $l\|x_t\|_{\mathcal{B}_b} \leq \|x_t\|_{\mathcal{B}_h} \leq l \sup_{0 \leq s \leq t} \|x(s)\|_{\mathcal{B}_b} + \|x_0\|_{\mathcal{B}_h}$, *where* $l = \int_{-\infty}^{0} h(s)ds < +\infty$. ∎

(B) If $x : (-\infty, \sigma + b] \to X, b > 0$, is such that $x_\sigma \in \mathcal{B}_h$ and $x|_{[\sigma, \sigma+b]} \in \mathcal{B}_b \subset \mathcal{PC}([\sigma, \sigma + b]; X)$, then for every $t \in [\sigma, \sigma + b]$ the following conditions hold:
(i) $x_t \in \mathcal{B}_h$,
(ii) $\|x(t)\| \leq H\|x_t\|_{\mathcal{B}_h}$,
(iii) $\|x_t\|_{\mathcal{B}_h} \leq K(t - \sigma)sup\{\|x(s)\| : \sigma \leq s \leq t\} + M(t - \sigma)\|x_\sigma\|_{\mathcal{B}_h}$,

where $H > 0$ is a constant; $K, M : [0, \infty) \to [1, \infty)$, K is continuous, M is locally bounded and H, K, M are independent of $x(.)$.

The paper has four sections. Section 2 deals with preliminaries which is useful for further investigation. We discuss the controllability result of system (1) in Section 3. We provide example to illustrate our theory in Section 4.

The terminology and notations are those generally used in functional analysis. In particular, $\mathcal{L}(X, Y)$ stands for the Banach space of bounded linear operators form X into Y and we abbreviate this notation to $\mathcal{L}(X)$ when $X = Y$. Moreover, $E_r(x, Z)$ denotes the closed ball with center at x and radius $r > 0$ in a metric space Z and, for a bounded function $\xi : [0, b] \to [0, \infty)$ we employ the notation ξ_t for

$$\xi_t = sup\{\xi(s) : s \in [0, t]\};\ 0 \leq t \leq b.$$

We shall prove main result using following theorem ([4],Theorem 6.5.4).

THEOREM 1.3 *Let* D *be a convex subset of a Banach space* X *and assume that* $0 \in D$. *Let* $F : D \to D$ *be a completely continuous map. Then the set* $\{x \in D : x = \lambda F(x), for\ some\ 0 < \lambda < 1\}$ *is unbounded or the map* F *has a fixed point in* D.
∎

2 Preliminaries

Here, $A : D(A) \subseteq X \to X$ is the infinitesimal generator of an uniformly bounded analytic semigroup of linear operators $\{T(t)\}_{t\geq 0}$ on X such that $0 \in \rho(A)$ and we assume that M_1 is a constant such that $\|T(t)\| \leq M_1$ for every $t \in J$. Under these conditions it is possible to define the fractional power $(-A)^{\alpha}, 0 < \alpha \leq 1$, as a closed linear operator on its domain $D((-A)^{\alpha})$.
Furthermore, the subspace $D(A^{\alpha})$ is dense in X and the expression

$$\|x\|_{\alpha} = \|A^{\alpha}x\|; \quad x \in D(A^{\alpha})$$

defines a norm on $D(A^{\alpha})$. Hereafter we denote by X_{α}, the Banach space $D(A^{\alpha})$ normed with $\|x\|_{\alpha}$, then the following lemma holds ([10]).

LEMMA 2.1 *Let* $0 < \gamma \leq \eta \leq 1$. *Then the following properties hold:*
(i) X_{η} *is a Banach space and* $X_{\eta} \hookrightarrow X_{\gamma}$ *is continuous.*
(ii) The function $s \to (-A)^{\eta}T(s)$ *is continuous in the uniform operator topology on* $(0, \infty)$ *and there exists* $C_{\eta} > 0$ *such that* $\|A^{\eta}T(t)\| \leq \frac{C_{\eta}}{t^{\eta}}$; *for every* $t > 0$. ■

Throughout this paper (H1), (H2) and (H3) satisfy the following conditions:
(H1) The function $f : J \times \mathcal{B}_h \to X$ satisfies the following conditions:

(i) For every $x : (-\infty, b] \to X$ such that $x_0 \in \mathcal{B}_h$ and $x|_J \in \mathcal{B}_b$, the function $t \to f(t, x_t)$ is strongly measurable. i.e., $f(., x_t) : J \to X$ is strongly measurable.
(ii) For each $t \in J$, the function $f(t, .) : \mathcal{B}_h \to X$ is continuous.
(iii) There exist an integrable function $m_f : J \to [0, \infty)$ and a continuous nondecreasing function $W_f : [0, \infty) \to (0, \infty)$ such that

$$\|f(t, \psi)\| \leq m_f(t)W_f(\|\psi\|_{\mathcal{B}_h}); (t, \psi) \in J \times \mathcal{B}_h.$$

(H2) There exists $0 < \beta < 1$ such that $g(.)$ is $X_{\beta}-$ valued.
(i) For every $x : (-\infty, b] \to X$ such that $x_0 \in \mathcal{B}_h$ and $x|_J \in \mathcal{B}_b$, the function $t \to (-A)^{\beta}g(t, x_t)$ is continuous on J.
(ii) For each $t \in J$, the function $(-A)^{\beta}g(t, .) : \mathcal{B}_h \to X$ is continuous and there exist positive constants c_1 and c_2 such that $lc_1 < 1$ and

$$\|(-A)^{\beta}g(t, \psi)\| \ \leq c_1\|\psi\|_{\mathcal{B}_h} + c_2(t, \psi); (t, \psi) \in J \times \mathcal{B}_h,$$

where $l = \int_{-\infty}^{0} h(s)ds < +\infty$.
(H3) Let $W : L^2(J, U) \to X_{\alpha}$ be the linear operator defined by

$$Wu = \int_0^b T(b - s)Bu(s)ds$$

Then $W : L^2(J,U)/kerW \to X_\alpha$ induces a bounded invertible operator $\widetilde{W}^{-1}$ and there exists positive constants M_2 and M_3 such that and $\|B\| \leq M_2$ and $\|\widetilde{W}^{-1}\| \leq M_3$ (for construction of $\widetilde{W}^{-1}$, refer [11]).

REMARK 2.2 *Let $x : (-\infty, b] \to X$ be a function such that $x_0 \in \mathcal{B}_h$ and $x|_J \in \mathcal{B}_b$. If the function g(.) satisfies (H2), then from the continuity of the function $s \to AT(t-s)$ in the uniform operator topology on $[0,t)$, (see Lemma 1.1), and the estimate*

$$\|(-A)T(t-s)g(s,x_s)\| = \|(-A)^{1-\beta}T(t-s)(-A)^{\beta}g(s,x_s)\| \leq \frac{C_{1-\beta}Cte}{(t-s)^{(1-\beta)}}, \quad (1)$$

then it follows that $AT(t-s)g(s,x_s)$ is integrable on $[0,t)$, for every $t > 0$. ■

DEFINITION 2.3 *A function $x : (-\infty, b] \to X$ is a mild solution of the initial value problem (1) if $x_0 \in \phi,\ x(.)|_J \in \mathcal{B}_b$ and*

$$\begin{aligned} x(t) &= T(t)(\phi(0) + g(0,\phi)) - g(t,x_t) - \int_0^t AT(t-s)g(s,x_s)ds + \int_0^t T(t-s)Bu(s)ds \\ &+ \int_0^t T(t-s)f(s,x_s+y_s)ds + \sum_{0<t_k<t} T(t-t_k)I_k(x(t_k));\ t \in J. \end{aligned}$$

DEFINITION 2.4 *The system (1) is said to be controllable on J, if for every $x_0 \in X$, there exists a control $u \in L^2(J,U)$ such that the mild solution $x(.)$ of (1) satisfies $x(.) = x_1$ and the conditions $\Delta x(t_k) = I_k(x(t_k)); k = 1,2,...,m;$ with $x_0 = \phi \in \mathcal{B}_h$.*

Let $\mathcal{B}_b^0$ be the space $\mathcal{B}_b^0 = \{x \in (-\infty, b] \to X$ such that $x_0 = 0 \in \mathcal{B}_h$ and $x|_J \in \mathcal{PC}\}$. For any $x \in \mathcal{B}_b^0$, $\|x\|_b = \|x_0\|_{\mathcal{B}_h} + sup\{\|x(s)\| : 0 \leq s \leq b\} = sup\{\|x(s)\| : 0 \leq s \leq b\}$. Thus $(\mathcal{B}_b^0, \|.\|_b)$ is a Banach space endowed with the norm of the uniform convergence.

Also from Lemma 1.2, we have

$$\begin{aligned} \|x_t + y_t\|_{\mathcal{B}_h} &\leq \|x_t\|_{\mathcal{B}_h} + \|y_t\|_{\mathcal{B}_b} \\ &\leq l \sup_{0\leq s\leq t} \|x(s)\| + \|x_0\|_{\mathcal{B}_h} + l \sup_{0\leq s\leq t} \|y(s)\| + \|y_0\|_{\mathcal{B}_h} \\ &\leq l(r_1 + M_1\|\phi(0)\|) + \|\phi\|_{\mathcal{B}_h} = r'. \end{aligned}$$

Hereafter, $y : (-\infty, b] \to X$ is the function defined by $y_0 = \phi$ on $-\infty < t \leq 0$ and $y(t) = T(t)\phi(0)$ on J. Clearly, $\|y_b\|_{\mathcal{B}_h} \leq (M_b + K_bM_1H)\|\phi\|_{\mathcal{B}_h}$.

Now we are in condition to establish the following result.

3 Controllability Result

THEOREM 3.1 *Suppose (H1), (H2) and (H3) are satisfied and the following conditions hold:*
(a) For each $r > 0$ and all $\epsilon > 0$ there is a compact set $U_{\epsilon,r} \subseteq X$ such that $T(\epsilon)f(s,\psi) \in U_{\epsilon,r}$ for every $s \in J, \psi \in B_r(0,\mathcal{B}_h)$.
(b) There exist positive constants $L_g, L_k,\ k = 1,2,...,m$; such that

$$\|(-A)^{\beta}g(t,\psi_1) - (-A)^{\beta}g(t,\psi_2)\| \le L_g\|\psi_1 - \psi_2\|_{\mathcal{B}_h}, \psi_k \in \mathcal{B}_h$$

$$\|I_k(\psi_1) - I_k(\psi_2)\| \le L_k\|\psi_1 - \psi_2\|_{\mathcal{B}_h};\ \psi_k \in \mathcal{B}_h; k = 1,2,...,m.$$

Further,

$$(1 + M_1M_2M_3b)K_b\{L_g(\|(-A)^{-\beta\|} + \frac{b^{\beta}}{\beta}C_{1-\beta}) + M_1 \lim_{\epsilon\to\infty} inf \frac{W_f(\xi)}{\xi}\int_0^b m_f(s)ds + M_1\sum L_k\} < 1. \quad (1)$$

Then system (1) is controllable on J.

Proof. Using the hypothesis (H3) for an arbitrary function $x(.)$ and $x_1 \in X$, we define the control formally as

$$\begin{aligned} u(t) &= \widetilde{W^{-1}}\Big\{x_1 - T(b)g(0,\phi) + g(t,x_t + y_t) - \int_0^b AT(b-s)g(s,x_s + y_s)ds \\ &- \int_0^b T(b-s)f(s,x_s + y_s)ds - \sum_{0<t_k<t} T(t-t_k)I_k(x(t_k) + y(t_k))\Big\}(t). \end{aligned}$$

We show that, using the above control, the operator Γ define by

$$\Gamma x(t) = \begin{cases} 0; \quad t \le 0 \\ T(t)g(0,\phi) - g(t,x_t + y_t) - \int_0^t AT(t-s)g(s,x_s + y_s)ds + \int_0^t T(t-s)Bu(s)ds \\ \int_0^t T(t-s)f(s,x_s + y_s)ds + \sum_{0<t_k<t} T(t-t_k)I_k(x(t_k) + y(t_k));\ t \in J \end{cases}$$

has a fixed point $x(.)$. This fixed point is then a mild solution of system (1) which implies that the system is controllable on J.
It is clear from (H1), (H2), (H3) and Remark (2.2) that $T(t-s)f(s,x_s+y_s), AT(t-s)g(s,x_s+y_s)$ and $T(t-s)Bu(s)$ are integrable on [0,t), for every $t \in J$. Therefore Γ is well defined with values in $\mathcal{B}_b^0$. In addition, from the axioms of phase space, the

Lebsegue dominated convergence theorem, and the conditions (H1), (H2) and (b), we can show that Γ is continuous from $\mathcal{B}_b^0$ into $\mathcal{B}_b^0$.

Step 1: We claim that there exist a positive number r such that $\Gamma(B_r(0,\mathcal{B}_b^0)) \subseteq B_r(0,\mathcal{B}_b^0)$. If this is not true, then for each positive number r, there exist a function $x^r(.) \in B_r$, but $x^r \notin \Gamma(B_r)$. i.e., $\|\Gamma x^r(t^r)\| > r$ for some $t \in J$. This yields from (H1), (H2), (H3) that

$$
\begin{aligned}
r &< \|\Gamma x^r(t^r)\| \\
&\leq M_1\|g(0,\psi)\| + \|(-A)^{-\beta}\|L_gK_br + \|g(t^r,y_{t^r})\| + L_gK_brC_{1-\beta}\frac{b^\beta}{\beta} \\
&+ C_{1-\beta}\int_0^{t^r}\frac{\|(-A)^\beta g(s,y_s)\|}{(t^r-s)^{1-\beta}}ds + M_1W_f(K_br + \sup_{s\in J}\|y_s\|_{\mathcal{B}_h})\int_0^b m_f(s)ds \\
&+ M_1M_2M_3\int_0^{t^r}\{\|x_1\| + M_1\|g(0,\phi)\| + \|(-A)^{-\beta}\|L_gK_br + \|g(t^r,y_{t^r})\| \\
&+ L_gK_brC_{1-\beta}\frac{b^\beta}{\beta} + C_{1-\beta}\int_0^b\frac{\|(-A)^\beta g(s,y_s)\|}{(b-s)^{1-\beta}}ds + M_1W_f(K_br + \sup_{s\in J}\|y_s\|_{\mathcal{B}_h}) \\
&\quad \int_0^b m_f(s)ds + M_1\sum_{k=1}^m(L_kK_br + \|I_k(y_{t_k^r})\|)\}ds + M_1\sum_{k=1}^m(L_kK_br + \|I_k(y_{t_k^r})\|) \\
1 &\leq (1+M_1M_2M_3b)K_b\{L_g(\|(-A)^{-\beta}\| + C_{1-\beta}\frac{b^\beta}{\beta}) + M_1\lim_{\epsilon\to\infty}inf\frac{W_f(\xi)}{\xi}\int_0^b m_f(s)ds \\
&+ M_1\sum L_k\}
\end{aligned}
$$

which is absurd. Hence for some positive number $r, \Gamma(B_r) \subseteq B_r$.

Now we consider the decomposition $\Gamma = \Gamma_1 + \Gamma_2$; where $\Gamma_i(x) = 0$ on $(-\infty, 0]$; $i = 1, 2$; such that

$$
\begin{aligned}
(\Gamma_1x)(t) &= T(t)g(0,\phi) - g(t,x_t+y_t) - \int_0^t AT(t-s)g(s,x_s+y_s)ds \\
&+ \sum_{0<t_k<t}T(t-t_k)I_k(x(t_k)+y(t_k)) + \int_0^t T(t-s)Bu_x(s)ds, t\in J \\
(\Gamma_2x)(t) &= \int_0^t T(t-s)f(s,x_s+y_s)ds, t\in J.
\end{aligned}
$$

We shall prove that Γ_1 is a contraction, while Γ_2 is completely continuous operator.

Step 2: Γ_1 is a contraction.

Let $x, y \in B_{r.}$ Then for each $t \in J$, we have from (H1), (H2), (H3) that

$$
\|\Gamma_1(x) - \Gamma_2(y)\|_b \leq K_b[L_g(\|(-A)^{-\beta}\|) + C_{1-\beta}\frac{b^\beta}{\beta} + M_1\sum L_k]\|x-y\|_b
$$

$$
\begin{aligned}
&+ \quad M_1 M_2 M_3 \int_0^{t^r} K_b [L_g(\|(-A)^{-\beta}\|) + C_{1-\beta}\frac{b^\beta}{\beta} + M_1 \sum L_k]\|x-y\|_b \\
&\leq \quad (1 + M_1 M_2 M_3 b) K_b [L_g(\|(-A)^{-\beta}\|) + C_{1-\beta}\frac{b^\beta}{\beta} + M_1 \sum L_k]\|x-y\|_b
\end{aligned}
$$

In view of (1), we conclude that Γ_1 is a contraction.

Step 3: Γ_2 is completely continuous.

From the Ascoli-Arzela theorem, to check the compactness of Γ_2, we prove that $\Gamma_2(B_r)$ is equicontinuous and $\Gamma_2(B_r)(t)$ is precompact for $t \in J$. For any $x \in B_r$ and for $0 \leq t < t + \tau < b$, we have

$$
\begin{aligned}
& \|\Gamma_2 x(t+\tau) - \Gamma_2 x(t)\| \\
= \; & \| \int_0^{t+\tau} T(t+\tau-s) f(s, x_s + y_s) ds - \int_0^t T(t-s) f(s, x_s + y_s) ds \| \\
\leq \; & \int_t^{t+\tau} \|T(t+\tau-s) f(s, x_s + y_s)\| ds + \int_0^t \|[T(t+\tau-s) - T(t-s)] f(s, x_s + y_s)\| ds \\
\leq \; & M_1 m_f(t) \int_t^{t+\tau} W_f \|x_s + y_s\|_{\mathcal{B}_h} ds + M_1 \int_0^t \|[T(\tau) - I] f(s, x_s + y_s)\| ds
\end{aligned}
$$

Since the elements of second term on the right hand side are included in a compact set, it follows that $\|[T(\tau) - I] f(s, x_s + y_s)\| \to 0$ as $\tau \to 0$ uniformly for $s \in [0, b]$ and $x \in B_r$. This implies that, for any $\epsilon > 0$, there exists $\delta > 0$ such that $|[T(\tau) - I] f(s, x_s + y_s)| < \epsilon$ for $0 \leq \tau < \delta$ and all $x \in B_r$. So for $0 \leq \tau < \delta$ and all $x \in B_r$,

$$
\|\Gamma_1(x) - \Gamma_2(y)\|_b \leq M_1 m_f(t) \int_t^{t+\tau} W_f \|x_s + y_s\|_{\mathcal{B}_h} ds + M_1 b \epsilon.
$$

Therefore $\Gamma_2(B_r) \subset \mathcal{B}_b^0$ is equicontinuous from the right at τ. Similarly, we can prove that $\Gamma_2(B_r)$ is equicontinuous from the left at $\tau > 0$. Thus $\Gamma_2(B_r)$ is equicontinuous. Now we will show that $\Gamma_2(B_r)(t)$ is relatively compact for $t \in J$.

Here case $t = 0$ is trivial. Assume that $0 < 2\epsilon < t \leq b$ and let $U_{\epsilon,r}$ be the compact set as defined in condition (a). Since $T(.)$ is strongly continuous on $[0, b]$, it is obvious that $U_\epsilon = \{T(s)x : s \in [\epsilon, b], x \in U_{\epsilon,r}\}$ is relatively compact in X. So application of mean value theorem for Bochner integral gives us

$$
\begin{aligned}
(\Gamma_2 x)(t) \; &= \int_0^{t-2\epsilon} T(t-s-\epsilon) T(\epsilon) f(s, x_s + y_s) ds + \int_{t-2\epsilon}^t T(t-s) f(s, x_s + y_s) ds \\
&\in (t-2\epsilon)\overline{conv(U_\epsilon)} + B_{r'}(0, \mathcal{B}_h)
\end{aligned}
$$

for $x \in B_{r'}$, where $conv(U_\epsilon)$ denotes the convex hull of (U_ϵ), define in [13] and r' is as defined in Section 2.
Thus $\Gamma_2(B_r)(t)$ is relatively compact in X. Collecting the equicontinuity of $\Gamma_2(B_r)$, we claim that $\Gamma_2(B_r)$ is completely continuous.
So, by steps 1 to 3, we can conclude that $\Gamma = \Gamma_1 + \Gamma_2$ is a condensing operator on B_r. By Theorem (1.3), there exists a fixed point $x(.)$ for Γ on B_r. Hence the proof is complete.

4 Example

Consider the following semilinear neutral impulsive differential heat equation with unbounded delay:

$$\begin{cases} \frac{\partial}{\partial t}\left(v(t,x) + \int_{-\infty}^{t} e^{4(s-t)} v(s,x)ds\right) = \frac{\partial^2 v(t,x)}{\partial x^2} + \int_{-\infty}^{t} \mu_1(t,x,s-t) f_1(v(s,x))ds + Bu(t); \\ x \in [0,\pi], t \in [0,b], t \neq t_k; \\ I_k(v(t_k^-,x)) = v(t_k,x) - v(t_k^-,x) = \Delta v(t_k)(x) = \int_{-\infty}^{t_k} a_k(t_k - s)v(s,x)dx, k = 1,2,...,m; \\ v(t,0) = v(t,\pi) = 0,\ t \in [0,b]; \\ v(t,x) = \phi(t,x), -\infty \le t \le 0,\ x \in [0,\pi]. \end{cases} \quad (1)$$

Let $X = L^2[0,\pi]$ be endowed with usual norm $|.|_{L^2}$. Define $A : D(A) \subseteq X \to X$ by $A\omega = \omega''$, where

$$D(A) = \{\omega \in X : \omega, \omega' \text{are absolutely continuous}, \omega'' \in X, \omega(0) = \omega(\pi) = 0\}.$$

Then $A\omega = \Sigma n^2 < \omega, \omega_n > \omega_n, \omega \in D(A)$, where $\omega_n(s) = \sqrt{\frac{2}{\pi}} sinns, n = 1, 2, ...$ is the orthogonal set of eigen functions in A. Here A is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\ge 0}$ in X and is given by,

$$T(t)\omega = \sum_{n=1}^{\infty} exp(-n^2 t)(\omega, \omega_n)\omega_n, \omega \in X.$$

Assume that $B : U \to Y, U \in J$ is a bounded linear operator and the operator

$$Wu = \int_0^b T(b-s)Bu(s)ds$$

has a bounded invertible operator $\widetilde{W}^{-1}$ in $L^2(J,U)/kerW$. (The existence of $\widetilde{W}^{-1}$ and its boundedness is discussed in [1]).
For $(t,\phi) \in J \times \mathcal{B}_h$, where $\phi(\theta)(x) = \phi(\theta,x), (\theta,x) \in (-\infty,0] \times [0,\pi]$, let $v(t)(x) =$

$v(t,x)$, $g(t,\phi)(x) = \int_{-\infty}^{0} e^{4\theta}\phi(\theta)(x)d\theta$, $f(t,\phi)(x) = \int_{-\infty}^{0} \mu_1(t,x,\theta)f_1(\phi(\theta)(x))d\theta$.
Moreover, we assume that
(a) The functions $\mu_1(t,x,\theta) \geq 0$ is continuous in $J\times[0,\pi]\times(-\infty,0]$ and $\int_{-\infty}^{0} \mu_1(t,x,\theta)d\theta = p_1(t,x) < \infty$.
(b) The function $f_1(.)$ is continuous, $0 \leq f_1(v(\theta,x)) \leq \theta_1(\int_{-\infty}^{0} e^{2s}|v(s,x)|_{L^2}ds)$ for $(\theta,x) \in (-\infty,0]\times[0,\pi]$, where $\theta_1 : [0,\infty) \to (0,\infty)$ is continuous and nondecreasing. Under these assumptions, $g(t,\phi) \in \mathcal{B}_h$ and $f(t,\phi) \in \mathcal{B}_h$ (refer [8]).
Thus system (1) is the abstract formulation of the system (1) and hence controllable on J. ∎

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