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Hyers-Ulam Stability of Functional Equations in 2-Banach Spaces

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Abstract

In [J. Math. Anal. Appl. 376 (1) (2011) 193–202], W.-G. Park investigate the generalized Hyers-Ulam stability of the functional equations

$f(x+y)=f(x)+f(y)$, $2f\left(\frac{x+y}{2}\right)=f(x)+f(y)$ and $f(x+y)+f(x-y)=2f(x)+2f(y)$

in 2-Banach spaces.

In this paper, we also investigate the generalized Hyers-Ulam stability of the same equations in 2-Banach spaces with different assumptions from [J. Math. Anal. Appl. 376 (1) (2011) 193–202].

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1. Introduction and preliminaries

In 1940, S.M. Ulam [6] suggested the stability problem of functional equations concerning the stability of group homomorphisms.

In the next year, D.H. Hyers [4] gave a first (partial) affirmative answer to the question of Ulam for Banach spaces as follows: *If* $\delta > 0$ *and if* $f : \mathcal{E} \to \mathcal{F}$ *is a mapping between Banach spaces* $\mathcal{E}$ *and* $\mathcal{F}$ *satisfying*

$$\|f(x+y)-f(x)-f(y)\| \le \delta$$

for all $x, y \in \mathcal{E}$, then there is a unique additive mapping $A : \mathcal{E} \to \mathcal{F}$ such that $\|f(x) - A(x)\| \le \delta$ for all $x, y \in \mathcal{E}$.

Thereafter, we call that type the Hyers-Ulam stability.

In the 1960's, S. Gähler [1, 2, 3] introduced the concept of linear 2-normed spaces.

Definition 1.1. Let $\mathcal{X}$ be a linear space over $\mathbb{R}$ with $\dim \mathcal{X} > 1$ and let $\|\cdot,\cdot\| : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ be a function satisfying the following properties:
(a) $\|x, y\| = 0$ if and only if x and y are linearly dependent,
(b) $\|x, y\| = \|y, x\|$,
(c) $\|\alpha x, y\| = |\alpha|\|x, y\|$,
(d) $\|x, y + z\| \le \|x, y\| + \|x, z\|$
for all $x, y, z \in \mathcal{X}$ and $\alpha \in \mathbb{R}$. Then the mapping $\|\cdot,\cdot\|$ is called a 2-*norm* on $\mathcal{X}$ and the pair $(\mathcal{X}, \|\cdot,\cdot\|)$ is called a *linear* 2-*normed space*. Sometimes the condition (d) called the *triangle inequality*.

In 2011, W.-G. Park [5] introduce a basic property of linear 2-normed spaces as follows.

Lemma 1.2. Let $(\mathcal{X}, \|\cdot,\cdot\|)$ be a linear 2-normed space. If $\|x, y\| = 0$ for all $y \in \mathcal{X}$, then $x = 0$.

For a linear 2-normed space $(\mathcal{X}, \|\cdot,\cdot\|)$, the function $x \to \|x, y\|$ is a continuous function of $\mathcal{X}$ into $\mathbb{R}$ for each fixed $y \in \mathcal{X}$ as follows.

Remark 1.3. Let $(\mathcal{X}, \|\cdot,\cdot\|)$ be a linear 2-normed space. Note that the conditions (a) and (d) implies that

$$\|x + y, z\| \le \|x, z\| + \|y, z\|$$

for all $x, y, z \in \mathcal{X}$. Putting $w := x + y$, we get $\|w, z\| \le \|x, z\| + \|w - x, z\|$ for all $x, y, z \in \mathcal{X}$. So $\|w, z\| - \|x, z\| \le \|w - x, z\|$ for all $x, z, w \in \mathcal{X}$. Replacing w by x and x by w in the above inequality, we get $\|x, z\| - \|w, z\| \le \|x - w, z\|$ for all $x, z, w \in \mathcal{X}$. Thus we have

$$| \, \|x, z\| - \|y, z\| \, | \le \|x - y, z\| \tag{1}$$

for all $x, y, z \in \mathcal{X}$. Hence the function $x \to \|x, y\|$ is a continuous function of $\mathcal{X}$ into $\mathbb{R}$ for each fixed $y \in \mathcal{X}$.

In the 1960's, S. Gähler and A. White [3, 7, 8] introduced the concept of 2-Banach spaces. In order to define completeness, the concepts of Cauchy sequences and convergence are required.

Definition 1.4. A sequence $\{x_n\}$ in a linear 2-normed space $\mathcal{X}$ is called a *Cauchy sequence* if there are two points $y, z \in \mathcal{X}$ such that y and z are linearly independent, $\lim_{m,n\to\infty} \|x_n - x_m, y\| = 0$ and $\lim_{m,n\to\infty} \|x_n - x_m, z\| = 0$.

Definition 1.5. A sequence $\{x_n\}$ in a linear 2-normed space $\mathcal{X}$ is called a *convergent sequence* if there is an $x \in \mathcal{X}$ such that $\lim_{n\to\infty} \|x_n - x, y\| = 0$ for all $y \in \mathcal{X}$. If $\{x_n\}$ converges to x, write $x_n \to x$ as $n \to \infty$ and call x the limit of $\{x_n\}$. In this case, we also write $\lim_{n\to\infty} x_n = x$.

Triangle inequality implies the following lemma (see [5]).

Lemma 1.6. For a convergent sequence $\{x_n\}$ in a linear 2-normed space $\mathcal{X}$, $\lim_{n\to\infty} \|x_n, y\| = \|\lim_{n\to\infty} x_n, y\|$ for all $y \in \mathcal{X}$.

Definition 1.7. A linear 2-normed space in which every Cauchy sequence is a convergent sequence is called a 2-*Banach space*.

In 2011, W.-G. Park [5] investigate approximate additive mappings, approximate Jensen mappings and approximate quadratic mappings in 2-Banach spaces.

In this paper, we also investigate the same kinds of mappings in 2-Banach spaces with different assumptions from [5].

2. Approximate additive mappings

Throughout this paper, let $\mathcal{X}$ be a normed linear space and $\mathcal{Y}$ a 2-Banach space.

In this section, we investigate the generalized Hyers-Ulam stability of the Cauchy functional equation in 2-Banach spaces with different assumptions from [5].

Theorem 2.1. *Let $\eta, \theta \in [0,\infty)$ and $p, q \in (0,1)$ and let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying*

$$\|f(x+y) - f(x) - f(y), z\| \leq \eta\|x\|^p + \theta\|y\|^q \tag{2}$$

for all $x, y \in \mathcal{X}$ and all $z \in \mathcal{Y}$. Then there is a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - A(x), y\| \leq \frac{\eta\|x\|^p}{2-2^p} + \frac{\theta\|x\|^q}{2-2^q} \tag{3}$$

for all $x \in \mathcal{X}$ and all $y \in \mathcal{Y}$.

Proof. Putting $y = x$ in (2), we get $\|f(2x) - 2f(x), z\| \leq \eta\|x\|^p + \theta\|x\|^q$ for all $x \in \mathcal{X}$ and all $z \in \mathcal{Y}$. So we get $\left\|f(x) - \frac{1}{2}f(2x), z\right\| \leq \frac{1}{2}(\eta\|x\|^p + \theta\|x\|^q)$ for all $x \in \mathcal{X}$ and all $z \in \mathcal{Y}$. Replacing x by $2^j x$ and dividing 2^j, we obtain

$$\left\|\frac{1}{2^j}f(2^j x) - \frac{1}{2^{j+1}}f(2^{j+1}x), z\right\| \leq \frac{1}{2^{j+1}}(\eta 2^{pj}\|x\|^p + \theta 2^{qj}\|x\|^q)$$

for all $x \in \mathcal{X}$, all $z \in \mathcal{Y}$ and all integers $j \geq 0$. For all integers l, m with $0 \leq l < m$, we get

$$\begin{aligned}\left\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x), z\right\| &\leq \sum_{j=l}^{m-1} \frac{1}{2^{j+1}}\left(\eta 2^{pj}\|x\|^p + \theta 2^{qj}\|x\|^q\right) \\ &= \frac{2^{(p-1)l} - 2^{(p-1)m}}{2-2^p}\eta\|x\|^p + \frac{2^{(q-1)l} - 2^{(q-1)m}}{2-2^q}\theta\|x\|^q \end{aligned} \tag{4}$$

for all $x \in \mathcal{X}$ and all $z \in \mathcal{Y}$. So we get $\lim_{l,m\to\infty}\left\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x), z\right\| = 0$ for all $x \in \mathcal{X}$ and all $z \in \mathcal{Y}$. Thus the sequence $\left\{\frac{1}{2^j}f(2^j x)\right\}$ is a Cauchy sequence in $\mathcal{Y}$. Since $\mathcal{Y}$ is a 2-Banach space, the sequence $\left\{\frac{1}{2^j}f(2^j x)\right\}$ converges. So one can define the mapping $A : \mathcal{X} \to \mathcal{Y}$ by $A(x) := \lim_{j\to\infty}\frac{1}{2^j}f(2^j x)$ for all $x \in \mathcal{X}$. That is, $\lim_{j\to\infty}\left\|\frac{1}{2^j}f(2^j x) - A(x), y\right\| = 0$ for all $x \in \mathcal{X}$ and all $y \in \mathcal{Y}$.

By Lemma 1.6 and (2), we get

$$\begin{aligned}
&\|A(x+y) - A(x) - A(y), z\| \\
&\qquad = \lim_{j\to\infty}\left\|\frac{1}{2^j}f(2^j x + 2^j y) - \frac{1}{2^j}f(2^j x) - \frac{1}{2^j}f(2^j y), z\right\| \\
&\qquad = \lim_{j\to\infty}\frac{1}{2^j}\left\|f(2^j x + 2^j y) - f(2^j x) - f(2^j y), z\right\| \\
&\qquad \le \eta\|x\|^p \lim_{j\to\infty} 2^{(p-1)j} + \theta\|y\|^q \lim_{j\to\infty} 2^{(q-1)j} = 0
\end{aligned}$$

for all $x, y \in \mathcal{X}$ and all $z \in \mathcal{Y}$. By Lemma 1.2, $A(x+y) - A(x) - A(y) = 0$ for all $x, y \in \mathcal{X}$. By Lemma 1.6 and (4), we have

$$\|f(x) - A(x), y\| = \lim_{m\to\infty}\left\|f(x) - \frac{1}{2^m}f(2^m x), y\right\| \le \frac{\eta\|x\|^p}{2-2^p} + \frac{\theta\|x\|^q}{2-2^q}$$

for all $x \in \mathcal{X}$ and all $y \in \mathcal{Y}$.

Now, let $B : \mathcal{X} \to \mathcal{Y}$ be another additive mapping satisfying (3). Then we have

$$\begin{aligned}
\|A(x) - B(x), y\| &= \frac{1}{2^j}\left\|A(2^j x) - B(2^j x), y\right\| \\
&\le \frac{1}{2^j}\left(\left\|A(2^j x) - f(2^j x), y\right\| + \left\|f(2^j x) - B(2^j x), y\right\|\right) \\
&\le \frac{\eta\|x\|^p}{1-2^{p-1}}2^{(p-1)j} + \frac{\theta\|x\|^q}{1-2^{q-1}}2^{(q-1)j},
\end{aligned}$$

which tends to zero as $j \to \infty$ for all $x \in \mathcal{X}$ and all $y \in \mathcal{Y}$. By Lemma 1.2, we can conclude that $A(x) = B(x)$ for all $x \in \mathcal{X}$. This proves the uniqueness of A. □

Theorem 2.2. *Let $\eta, \theta \in [0, \infty)$ and $p, q \in (1, \infty)$ and let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying*

$$\|f(x+y) - f(x) - f(y), z\| \le \eta\|x\|^p + \theta\|y\|^q \tag{5}$$

for all $x, y \in \mathcal{X}$ and all $z \in \mathcal{Y}$. Then there is a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that $\|f(x) - A(x), y\| \le \frac{\eta\|x\|^p}{2^p-2} + \frac{\theta\|x\|^q}{2^q-2}$ for all $x \in \mathcal{X}$ and all $y \in \mathcal{Y}$.

Proof. Putting $y = x$ in (5), we get $\|f(2x) - 2f(x), z\| \le \eta\|x\|^p + \theta\|x\|^q$ for all $x \in \mathcal{X}$ and all $z \in \mathcal{Y}$. So we get $\left\|f(x) - 2f\left(\frac{x}{2}\right), z\right\| \le \frac{\eta}{2^p}\|x\|^p + \frac{\theta}{2^q}\|x\|^q$ for all $x \in \mathcal{X}$ and all $z \in \mathcal{Y}$. Replacing x by $\frac{x}{2^j}$ and multiplying 2^j, we obtain

$$\left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right), z\right\| \le 2^j \left(\frac{\eta}{2^{p(j+1)}}\|x\|^p + \frac{\theta}{2^{q(j+1)}}\|x\|^q\right)$$

for all $x \in \mathcal{X}$, all $z \in \mathcal{Y}$ and all integers $j \ge 0$. For all integers l, m with $0 \le l < m$, we get

$$\begin{aligned}\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right), z\right\| &\le \sum_{j=l}^{m-1} 2^j \left(\frac{\eta}{2^{p(j+1)}}\|x\|^p + \frac{\theta}{2^{q(j+1)}}\|x\|^q\right) \\ &= \frac{1}{2^p-2}\left(\frac{1}{2^{(p-1)l}} - \frac{1}{2^{(p-1)m}}\right)\eta\|x\|^p + \frac{1}{2^q-2}\left(\frac{1}{2^{(q-1)l}} - \frac{1}{2^{(q-1)m}}\right)\theta\|x\|^q\end{aligned}$$

for all $x \in \mathcal{X}$ and all $z \in \mathcal{Y}$. So we get $\lim_{l,m\to\infty} \left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right), z\right\| = 0$ for all $x \in \mathcal{X}$ and all $z \in \mathcal{Y}$. Thus the sequence $\left\{2^j f\left(\frac{x}{2^j}\right)\right\}$ is a Cauchy sequence in $\mathcal{Y}$. Since $\mathcal{Y}$ is a 2-Banach space, the sequence $\left\{2^j f\left(\frac{x}{2^j}\right)\right\}$ converges. So one can define the mapping $A : \mathcal{X} \to \mathcal{Y}$ by $A(x) := \lim_{j\to\infty} 2^j f\left(\frac{x}{2^j}\right)$ for all $x \in \mathcal{X}$. That is, $\lim_{j\to\infty} \left\|2^j f\left(\frac{x}{2^j}\right) - A(x), y\right\| = 0$ for all $x \in \mathcal{X}$ and all $y \in \mathcal{Y}$.

The further part of the proof is similar to the proof of Theorem 2.1. □

3. Approximate Jensen mappings

In this section, we investigate the generalized Hyers-Ulam stability of the Jensen functional equation in 2-Banach spaces with different assumptions from [5].

Theorem 3.1. *Let $\eta, \theta \in [0, \infty)$ and $p, q \in (0, 1)$ and let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying*

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y), z\right\| \le \eta\|x\|^p + \theta\|y\|^q \tag{6}$$

for all $x, y \in \mathcal{X}$ and all $z \in \mathcal{Y}$. Then there is a unique Jensen mapping $J : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - J(x), y\| \le \frac{2\eta}{3-3^p}\|x\|^p + \theta\frac{1+3^q}{3-3^q}\|x\|^q \tag{7}$$

for all $x \in \mathcal{X}$ and all $y \in \mathcal{Y}$.

Proof. Define a mapping $g : \mathcal{X} \to \mathcal{Y}$ by $g(x) := f(x) - f(0)$ for all $x \in \mathcal{X}$. Then $g(0) = 0$. Since f satisfies (6), g also satisfies (6). That is, g satisfies the inequality

$$\left\|2g\left(\frac{x+y}{2}\right) - g(x) - g(y), z\right\| \le \eta\|x\|^p + \theta\|y\|^q \tag{8}$$

for all $x, y \in \mathcal{X}$ and all $z \in \mathcal{Y}$. Letting $y = -x$ in (8), we get

$$\|g(x) + g(-x), z\| \le \eta\|x\|^p + \theta\|x\|^q$$

for all $x \in \mathcal{X}$ and all $z \in \mathcal{Y}$. Replacing x by $-x$ and y by $3x$ in (8), we get

$$\|2g(x) - g(-x) - g(3x), z\| \le \eta\|x\|^p + \theta 3^q\|x\|^q$$

for all $x \in \mathcal{X}$ and all $z \in \mathcal{Y}$. By the above two inequalities and the triangular inequality, we get

$$\begin{aligned}\|3g(x) - g(3x), z\| &\le \|g(x) + g(-x), z\| + \|2g(x) - g(-x) - g(3x), z\| \\ &\le 2\eta\|x\|^p + \theta(1+3^q)\|x\|^q\end{aligned}$$

for all $x \in \mathcal{X}$ and all $z \in \mathcal{Y}$. So we get

$$\left\|g(x) - \frac{1}{3}g(3x), z\right\| \le \frac{1}{3}\left(2\eta\|x\|^p + \theta(1+3^q)\|x\|^q\right)$$

for all $x \in \mathcal{X}$ and all $z \in \mathcal{Y}$. Replacing x by $3^j x$ and dividing 3^j, we obtain

$$\left\|\frac{1}{3^j}g(3^j x) - \frac{1}{3^{j+1}}g(3^{j+1}x), z\right\| \le \frac{1}{3^{j+1}}\left(2\cdot 3^{pj}\eta\|x\|^p + \theta(1+3^q)3^{qj}\|x\|^q\right)$$

for all $x \in \mathcal{X}$, all $z \in \mathcal{Y}$ and all integers $j \ge 0$. For all integers l, m with $0 \le l < m$, we get

$$\begin{aligned}\left\|\frac{1}{3^l}g(3^l x) - \frac{1}{3^m}g(3^m x), z\right\| &\le \sum_{j=l}^{m-1}\frac{1}{3^{j+1}}\left(2\cdot 3^{pj}\eta\|x\|^p + \theta(1+3^q)3^{qj}\|x\|^q\right) \\ &= 2\eta\frac{3^{(p-1)l} - 3^{(p-1)m}}{3-3^p}\|x\|^p + \theta(1+3^q)\frac{3^{(q-1)l} - 3^{(q-1)m}}{3-3^q}\|x\|^q \qquad (9)\end{aligned}$$

for all $x \in \mathcal{X}$ and all $z \in \mathcal{Y}$. So we get $\lim_{l,m\to\infty}\left\|\frac{1}{3^l}f(3^l x) - \frac{1}{3^m}g(3^m x), z\right\| = 0$ for all $x \in \mathcal{X}$ and all $z \in \mathcal{Y}$. Thus the sequence $\left\{\frac{1}{3^j}g(3^j x)\right\}$ is a Cauchy sequence in $\mathcal{Y}$. Since $\mathcal{Y}$ is a 2-Banach space, the sequence $\left\{\frac{1}{3^j}g(3^j x)\right\}$ converges. So one can define the mapping $A : \mathcal{X} \to \mathcal{Y}$ by

$$A(x) := \lim_{j\to\infty}\frac{1}{3^j}g(3^j x) = \lim_{j\to\infty}\frac{1}{3^j}f(3^j x)$$

for all $x \in \mathcal{X}$. That is, $\lim_{j\to\infty}\left\|\frac{1}{3^j}f(3^j x) - A(x), y\right\| = 0$ for all $x \in \mathcal{X}$ and all $y \in \mathcal{Y}$. By Lemma 1.6 and (9), we have

(10)

$$\|g(x) - A(x), y\| = \lim_{m\to\infty}\left\|g(x) - \frac{1}{3^m}g(3^m x), y\right\| \le \frac{2\eta}{3-3^p}\|x\|^p + \theta\frac{1+3^q}{3-3^q}\|x\|^q$$

for all $x \in \mathcal{X}$ and all $y \in \mathcal{Y}$.

Define a mapping $J : \mathcal{X} \to \mathcal{Y}$ by $J(x) := A(x) + f(0)$ for all $x \in \mathcal{X}$. Then we have $f(x) - J(x) = g(x) - A(x)$ and

$$\|f(x) - J(x), y\| = \|g(x) - A(x), y\| \le \frac{2\eta}{3-3^p}\|x\|^p + \theta\frac{1+3^q}{3-3^q}\|x\|^q$$

for all $x \in \mathcal{X}$ and all $y \in \mathcal{Y}$. By Lemma 1.6, we get

$$\begin{aligned}\left\|3^k A(x) - A(3^k x), y\right\| &= \lim_{j\to\infty} \left\|\frac{3^k}{3^j} f(3^j x) - \frac{1}{3^j} f(3^{j+k}x), y\right\| \\ &= 3^k \lim_{j\to\infty} \left\|\frac{1}{3^j} f(3^j x) - \frac{1}{3^{j+k}} f(3^{j+k}x), y\right\| = 3^k \|A(x) - A(x), y\| = 0 \end{aligned} \tag{11}$$

for all $x \in \mathcal{X}$, all $y \in \mathcal{Y}$ and all $k \in \mathbb{N}$. By Lemma 1.2 and (11), we get

$$3^k A(x) = A(3^k x) \tag{12}$$

for all $x \in \mathcal{X}$ and all $k \in \mathbb{N}$. Putting $x = 0$ and $k = 1$ in (12), we get $3A(0) = A(0)$, that is, $A(0) = 0$. By (8), (10) and (12), we have

$$\begin{aligned}&\|2A(2x) - 4A(x), y\| = \|2A(2x) - A(3x) - A(x), y\| \\ &\quad = \frac{1}{3^k}\left\|2A\big(3^k \cdot 2x\big) - A\big(3^k \cdot 3x\big) - A\big(3^k x\big), y\right\| \\ &\quad \le \frac{1}{3^k}\bigg[\left\|2A\big(2\cdot 3^k x\big) - 2g\big(2\cdot 3^k x\big), y\right\| + \left\|A\big(3^{k+1}x\big) - g\big(3^{k+1}x\big), y\right\| \\ &\qquad + \left\|A\big(3^k x\big) - g\big(3^k x\big), y\right\| + \left\|2g\left(\frac{3^{k+1}x + 3^k x}{2}\right) - g\big(3^{k+1}x\big) - g\big(3^k x\big), y\right\|\bigg] \\ &\quad \le \frac{\eta 3^{(p-1)k}}{3 - 3^p}\big[2(1 + 2^{p+1}) + 3^p(5 - 3^p)\big]\|x\|^p \\ &\qquad + \frac{\theta 3^{(q-1)k}}{3 - 3^q}\big[(5 + 2^{q+2})(1 + 3^q) - 2\cdot 3^q(1 - 3^q)\big]\|x\|^q,\end{aligned}$$

which tends to zero as $k \to \infty$ for all $x \in \mathcal{X}$ and all $y \in \mathcal{Y}$. By Lemma 1.2, we obtain $A(2x) = 2A(x)$ for all $x \in \mathcal{X}$. By (8) and (12), we get

$$\begin{aligned}&\left\|2A\Big(\frac{x+y}{2}\Big) - A(x) - A(y), z\right\| \\ &\qquad = \lim_{j\to\infty} \frac{1}{3^j}\left\|g\left(\frac{3^j x + 3^j y}{2}\right) - g(3^j x) - g(3^j y), z\right\| \\ &\qquad \le \eta\|x\|^p \lim_{j\to\infty} 3^{(p-1)j} + \theta\|y\|^q \lim_{j\to\infty} 3^{(q-1)j} = 0\end{aligned}$$

for all $x, y \in \mathcal{X}$ and all $z \in \mathcal{Y}$. By Lemma 1.6, A is a Jensen mapping. Thus we get

$$2J\Big(\frac{x+y}{2}\Big) = 2A\Big(\frac{x+y}{2}\Big) + 2f(0) = A(x) + A(y) + 2f(0) = J(x) + J(y)$$

for all $x, y \in \mathcal{X}$. Hence J is also a Jensen mapping.

Now, let $K:\mathcal{X}\to\mathcal{Y}$ be another Jensen mapping satisfying (7). Then we have

$$\begin{aligned}\|J(x)-K(x),y\| &= \frac{1}{3^j}\left\|J(3^jx)-K(3^jx),y\right\|\\ &\le \frac{1}{3^j}\left(\left\|J(3^jx)-f(3^jx),y\right\|+\left\|f(3^jx)-K(3^jx),y\right\|\right)\\ &\le \frac{2\eta 3^{(p-1)j}}{3-3^p}\|x\|^p+\theta\frac{1+3^q}{1-3^q}3^{(q-1)j}\|x\|^q,\end{aligned}$$

which tends to zero as $j\to\infty$ for all $x\in\mathcal{X}$ and all $y\in\mathcal{Y}$. By Lemma 1.2, we obtain that $J(x)=K(x)$ for all $x\in\mathcal{X}$. This proves the uniqueness of J. □

Theorem 3.2. *Let $\eta,\theta\in[0,\infty)$ and $p,q\in(1,\infty)$ and let $f:\mathcal{X}\to\mathcal{Y}$ be a mapping satisfying $f(0)=0$ such that*

$$\left\|2f\left(\frac{x+y}{2}\right)-f(x)-f(y),z\right\|\le\eta\|x\|^p+\theta\|y\|^q$$

for all $x,y\in\mathcal{X}$ and all $z\in\mathcal{Y}$. Then there is a unique additive mapping $A:\mathcal{X}\to\mathcal{Y}$ such that $\|f(x)-A(x),y\|\le\frac{2\eta}{3^p-3}\|x\|^p+\theta\frac{3^q+1}{3^q-3}\|x\|^q$ for all $x\in\mathcal{X}$ and all $y\in\mathcal{Y}$.

Proof. By the same argument as in the proof of Theorem 3.1, we get

$$\begin{aligned}\|3f(x)-f(3x),z\| &\le \|f(x)+f(-x),z\|+\|2f(x)-f(-x)-f(3x),z\|\\ &\le 2\eta\|x\|^p+\theta(1+3^q)\|x\|^q\end{aligned}$$

for all $x\in\mathcal{X}$ and all $z\in\mathcal{Y}$. So we get $\left\|f(x)-3f\left(\frac{x}{3}\right),z\right\|\le\frac{2\eta}{3^p}\|x\|^p+\theta\frac{1+3^q}{3^q}\|x\|^q$ for all $x\in\mathcal{X}$ and all $z\in\mathcal{Y}$. Replacing x by $\frac{x}{3^j}$ and multiplying 3^j, we obtain

$$\left\|3^jf\left(\frac{x}{3^j}\right)-3^{j+1}f\left(\frac{x}{3^{j+1}}\right),z\right\|\le 3^j\left(\frac{2\eta}{3^{p(j+1)}}\|x\|^p+\theta\frac{1+3^q}{3^{q(j+1)}}\|x\|^q\right)$$

for all $x\in\mathcal{X}$, all $z\in\mathcal{Y}$ and all integers $j\ge0$. For all integers l,m with $0\le l<m$, we get

$$\begin{aligned}\left\|3^lf\left(\frac{x}{3^l}\right)-3^mf\left(\frac{x}{3^m}\right),z\right\| &\le \sum_{j=l}^{m-1}3^j\left(\frac{2\eta}{3^{p(j+1)}}\|x\|^p+\theta\frac{1+3^q}{3^{q(j+1)}}\|x\|^q\right)\\ &= \frac{2\eta}{3^p-3}\left(\frac{1}{3^{(p-1)l}}-\frac{1}{3^{(p-1)m}}\right)\|x\|^p+\theta\frac{1+3^q}{3^q-3}\left(\frac{1}{3^{(q-1)l}}-\frac{1}{3^{(q-1)m}}\right)\|x\|^q\end{aligned}$$

for all $x\in\mathcal{X}$ and all $z\in\mathcal{Y}$. So we get $\lim_{l,m\to\infty}\left\|3^lf\left(\frac{x}{3^l}\right)-3^mf\left(\frac{x}{3^m}\right),z\right\|=0$ for all $x\in\mathcal{X}$ and all $z\in\mathcal{Y}$. Thus the sequence $\left\{3^jf\left(\frac{x}{3^j}\right)\right\}$ is a Cauchy sequence in $\mathcal{Y}$. Since $\mathcal{Y}$ is a 2-Banach space, the sequence $\left\{3^jf\left(\frac{x}{3^j}\right)\right\}$ converges. So one can define the mapping $A:\mathcal{X}\to\mathcal{Y}$ by $A(x):=\lim_{j\to\infty}3^jf\left(\frac{x}{3^j}\right)$ for all $x\in\mathcal{X}$. That is, $\lim_{j\to\infty}\left\|3^jf\left(\frac{x}{3^j}\right)-A(x),y\right\|=0$ for all $x\in\mathcal{X}$ and all $y\in\mathcal{Y}$.

The further part of the proof is similar to the proof of Theorem 3.1. □

4. Approximate quadratic mappings

In this section, we investigate the generalized Hyers-Ulam stability of the quadratic functional equation in 2-Banach spaces with different assumptions from [5].

Theorem 4.1. *Let $\eta, \theta \in [0, \infty)$ and $p, q \in (0, 2)$ and let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y), z\| \leq \eta\|x\|^p + \theta\|y\|^q \tag{13}$$

for all $x, y \in \mathcal{X}$ and all $z \in \mathcal{Y}$. Then there is a unique quadratic mapping $Q : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - Q(x), y\| \leq \frac{\eta\|x\|^p}{4 - 2^p} + \frac{\theta\|x\|^q}{4 - 2^q} + \frac{1}{3}\|f(0), y\| \tag{14}$$

for all $x \in \mathcal{X}$ and all $y \in \mathcal{Y}$.

Proof. Putting $y = x$ in (13), we get $\|f(2x) - 4f(x) + f(0), z\| \leq \eta\|x\|^p + \theta\|x\|^q$ for all $x \in \mathcal{X}$ and all $z \in \mathcal{Y}$. So we get

$$\left\|f(x) - \frac{1}{4}f(2x) - \frac{1}{4}f(0), z\right\| \leq \frac{1}{4}(\eta\|x\|^p + \theta\|x\|^q)$$

for all $x \in \mathcal{X}$ and all $z \in \mathcal{Y}$. Replacing x by $2^j x$ and dividing 4^j, we obtain

$$\left\|\frac{1}{4^j}f(2^j x) - \frac{1}{4^{j+1}}f(2^{j+1}x) - \frac{1}{4^{j+1}}f(0), z\right\| \leq \frac{1}{4^{j+1}}(\eta 2^{pj}\|x\|^p + \theta 2^{qj}\|x\|^q)$$

for all $x \in \mathcal{X}$, all $z \in \mathcal{Y}$ and all integers $j \geq 0$. For all integers l, m with $0 \leq l < m$, we get

$$\begin{aligned}
&\left\|\frac{1}{4^l}f(2^l x) - \frac{1}{4^m}f(2^m x) - \frac{1}{3}\left(\frac{1}{4^l} - \frac{1}{4^m}\right)f(0), z\right\| \\
&\qquad \leq \sum_{j=l}^{m-1} \frac{1}{4^{j+1}}\left(\eta 2^{pj}\|x\|^p + \theta 2^{qj}\|x\|^q\right) \\
&\qquad = \frac{2^{(p-2)l} - 2^{(p-2)m}}{4 - 2^p}\eta\|x\|^p + \frac{2^{(q-2)l} - 2^{(q-2)m}}{4 - 2^q}\theta\|x\|^q
\end{aligned} \tag{15}$$

for all $x \in \mathcal{X}$ and all $z \in \mathcal{Y}$. So we get $\lim_{l,m\to\infty} \left\|\frac{1}{4^l}f(2^l x) - \frac{1}{4^m}f(2^m x), z\right\| = 0$ for all $x \in \mathcal{X}$ and all $z \in \mathcal{Y}$. Thus the sequence $\left\{\frac{1}{4^j}f(2^j x)\right\}$ is a Cauchy sequence in $\mathcal{Y}$. Since $\mathcal{Y}$ is a 2-Banach space, the sequence $\left\{\frac{1}{4^j}f(2^j x)\right\}$ converges. So one can define the mapping $Q : \mathcal{X} \to \mathcal{Y}$ by

$$Q(x) := \lim_{j\to\infty} \frac{1}{4^j}f(2^j x)$$

for all $x \in \mathcal{X}$. That is, $\lim_{j\to\infty} \left\|\frac{1}{4^j}f(2^j x) - Q(x), y\right\| = 0$ for all $x \in \mathcal{X}$ and all $y \in \mathcal{Y}$.

By Lemma 1.6 and (13), we get

$$\begin{aligned}
&\|Q(x+y)+Q(x-y)-2Q(x)-2Q(y),z\| \\
&\quad= \lim_{j\to\infty}\left\|\frac{1}{4^j}f(2^jx+2^jy)+\frac{1}{4^j}f(2^jx-2^jy)-\frac{2}{4^j}f(2^jx)-\frac{2}{4^j}f(2^jy),z\right\| \\
&\quad= \lim_{j\to\infty}\frac{1}{4^j}\big\|f(2^jx+2^jy)+f(2^jx-2^jy)-2f(2^jx)-2f(2^jy),z\big\| \\
&\quad\le \eta\|x\|^p\lim_{j\to\infty}2^{(p-2)j}+\theta\|y\|^q\lim_{j\to\infty}2^{(q-2)j}=0
\end{aligned}$$

for all $x,y\in\mathcal{X}$ and all $z\in\mathcal{Y}$. By Lemma 1.2, Q is a quadratic mapping. By Lemma 1.6 and (15), we have

$$\begin{aligned}
\left\|f(x)-Q(x)-\frac{1}{3}f(0),y\right\| &= \lim_{m\to\infty}\left\|f(x)-\frac{1}{4^m}f(2^mx)-\frac{1}{3}\left(1-\frac{1}{4^m}\right)f(0),y\right\| \\
&\le \frac{\eta\|x\|^p}{4-2^p}+\frac{\theta\|x\|^q}{4-2^q}
\end{aligned}$$

for all $x\in\mathcal{X}$ and all $y\in\mathcal{Y}$. By (1), we have

$$\|f(x)-Q(x),y\|\le\frac{\eta\|x\|^p}{4-2^p}+\frac{\theta\|x\|^q}{4-2^q}+\frac{1}{3}\|f(0),y\|$$

for all $x\in\mathcal{X}$ and all $y\in\mathcal{Y}$.

Now, let $R:\mathcal{X}\to\mathcal{Y}$ be another additive mapping satisfying (14). Then we have

$$\begin{aligned}
\|Q(x)-R(x),y\| &= \frac{1}{4^j}\big\|Q(2^jx)-R(2^jx),y\big\| \\
&\le \frac{1}{4^j}\big(\big\|Q(2^jx)-f(2^jx),y\big\|+\big\|f(2^jx)-R(2^jx),y\big\|\big) \\
&\le \frac{\eta\|x\|^p}{2-2^{p-1}}2^{(p-2)j}+\frac{\theta\|x\|^q}{2-2^{q-1}}2^{(q-2)j}+\frac{2}{3\cdot4^j}\|f(0),y\|,
\end{aligned}$$

which tends to zero as $j\to\infty$ for all $x\in\mathcal{X}$ and all $y\in\mathcal{Y}$. By Lemma 1.2, we can conclude that $Q(x)=R(x)$ for all $x\in\mathcal{X}$. This proves the uniqueness of Q. □

Theorem 4.2. *Let $\eta,\theta\in[0,\infty)$ and $p,q\in(2,\infty)$ and let $f:\mathcal{X}\to\mathcal{Y}$ be a mapping satisfying $f(0)=0$ such that*

(16) $$\|f(x+y)+f(x-y)-2f(x)-2f(y),z\|\le\eta\|x\|^p+\theta\|y\|^q$$

for all $x,y\in\mathcal{X}$ and all $z\in\mathcal{Y}$. Then there is a unique quadratic mapping $Q:\mathcal{X}\to\mathcal{Y}$ such that $\|f(x)-Q(x),y\|\le\frac{\eta\|x\|^p}{2^p-4}+\frac{\theta\|x\|^q}{2^q-4}$ for all $x\in\mathcal{X}$ and all $y\in\mathcal{Y}$.

Proof. Putting $y=x$ in (16), we get $\|f(2x)-4f(x),z\|\le\eta\|x\|^p+\theta\|x\|^q$ for all $x\in\mathcal{X}$ and all $z\in\mathcal{Y}$. So we get $\big\|f(x)-4f\big(\frac{x}{2}\big),z\big\|\le\frac{\eta}{2^p}\|x\|^p+\frac{\theta}{2^q}\|x\|^q$ for all $x \in \mathcal{X}$ and all $z \in \mathcal{Y}$. Replacing x by $\frac{x}{2^j}$ and multiplying 4^j, we obtain

$$\left\|4^j f\left(\frac{x}{2^j}\right) - 4^{j+1} f\left(\frac{x}{2^{j+1}}\right), z\right\| \leq 4^j \left(\frac{\eta}{2^{p(j+1)}}\|x\|^p + \frac{\theta}{2^{q(j+1)}}\|x\|^q\right)$$

for all $x \in \mathcal{X}$, all $z \in \mathcal{Y}$ and all integers $j \geq 0$. For all integers l, m with $0 \leq l < m$, we get

$$\begin{aligned}\left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right), z\right\| &\leq \sum_{j=l}^{m-1} 4^j \left(\frac{\eta}{2^{p(j+1)}}\|x\|^p + \frac{\theta}{2^{q(j+1)}}\|x\|^q\right) \\ &= \frac{2^{(2-p)l} - 2^{(2-p)m}}{2^p - 4}\eta\|x\|^p + \frac{2^{(2-q)l} - 2^{(2-q)m}}{2^q - 4}\theta\|x\|^q\end{aligned}$$

for all $x \in \mathcal{X}$ and all $z \in \mathcal{Y}$. So we get $\lim_{l,m\to\infty} \left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right), z\right\| = 0$ for all $x \in \mathcal{X}$ and all $z \in \mathcal{Y}$. Thus the sequence $\left\{4^j f\left(\frac{x}{2^j}\right)\right\}$ is a Cauchy sequence in $\mathcal{Y}$. Since $\mathcal{Y}$ is a 2-Banach space, the sequence $\left\{4^j f\left(\frac{x}{2^j}\right)\right\}$ converges. So one can define the mapping $Q : \mathcal{X} \to \mathcal{Y}$ by $Q(x) := \lim_{j\to\infty} 4^j f\left(\frac{x}{2^j}\right)$ for all $x \in \mathcal{X}$. That is, $\lim_{j\to\infty} \left\|4^j f\left(\frac{x}{2^j}\right) - Q(x), y\right\| = 0$ for all $x \in \mathcal{X}$ and all $y \in \mathcal{Y}$.

The further part of the proof is similar to the proof of Theorem 4.1. □

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