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Some Hermite-Hadamard-like Type Inequalities for Logarithmically Convex Functions

Jaekeun Park

Department of Mathematics
Hanseo University
Seosan, Choongnam, 356-706, Korea
jkpark@hanseo.ac.kr

Abstract

In this article, some Hermite-Hadamard-like type integral inequalities for logarithmically convex functions are obtained and the applications to the special means are given.

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1 Introduction

In many areas of mathematics and sciences, logarithmically convex(*log*-convex) functions are of interest. They have been found to play an important role in the theory of special functions and mathematical statistics(see, e.g., [6, 8, 9, 20, 21]).

Definition 1. A function $f : I \to R$ is said to be convex on an interval I in R if the following inequalty

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1}$$

holds, for all $x, y \in I$ and $t \in [0, 1]$.

One of the most famous inequalities for convex functions is Hermite-Hadamard type integral inequality. This double inequality is stated as follows: let f be a convex function on some nonempty interval $[a, b]$ of real line R, where $a \neq b$, then the following inequalities hold:

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}. \tag{2}$$

The Hermite-Hadamard type integral inequality for convex functions has received renewed attention in recent years and the remarkable varieties of refinements and generalizations have been found in [1]-[11] and [14, 15, 18, 22, 23].

Definition 2. A function $f : I \to R_+ = (0, \infty)$ is said to be log-convex or multiplicably convex on an interval I in R if $\log(f)$ is convex, or equivalently, if the following inequalty

$$f(tx + (1-t)y) \leq [f(x)]^t [f(y)]^{1-t} \tag{3}$$

holds, for all $x, y \in I$ and $t \in [0, 1]$.

We note that if f and g are convex functions and g is monotonic nondecreasing, then $g \circ f$ is convex. Moreover, since $f = \exp(\log f)$, it follows that a *log*-convex function is convex, but the converse is not true [17].

Definition 3. A function $f : I \to R_+ = (0, \infty)$ is said to be geometric-arithmetically convex or GA-convex on an interval I in R if the following inequalty

$$f(x^t y^{1-t}) \leq tf(x) + (1-t)f(y) \tag{4}$$

holds for all $x, y \in I$ and $t \in [0, 1]$, where $x^t y^{1-t}$ and $tf(x) + (1-t)f(y)$ are respectively called the weighted geometric mean of two positive numbers x and y and the weighted mean of $f(x)$ and $f(y)$.

For the properties and refinements of the geometric-arithmetically convex functions, see [12, 13, 16, 19, 24, 25]

If the above inequality (4) is reversed, then f is called logarithmically concave, or simply log-concave. Apparently, it would seem that log-concave (log-convex) functions would be unremarkable because they are simply related to concave (convex) functions. It is well-known that the product of log-concave (log-convex)functions is also log-concave (log-convex). Moreover the sum of log-convex functions is also log-convex. However, the sum of log-concave functions is not necessarily log-concave. Due to their interesting properties, the log-convex (log-concave) functions frequently appear in many problems of classical analysis and probablity theory.

In this article, we will use the following notations and conventions: For $I \subseteq R$ and $a, b \in I$ with $0 < a < b$,

$$A(a,b) = \frac{a+b}{2}, \quad G(a,b) = \sqrt{ab},$$
$$H(a,b) = \frac{2ab}{a+b}, \quad L(a,b) = \frac{b-a}{\ln b - \ln a},$$
$$P(a,b) = \frac{1}{3}\Big\{2G(a,b) + A(a,b)\Big\}.$$

are the arithmetic, geometric, harmonic, and logarithmic mean, respectively.

Note that

$$\min\big\{a,b\big\} \leq H(a,b) \leq G(a,b) \leq L(a,b) \leq P(a,b) \leq A(a,b) \leq \max\{a,b\}.$$

In [5], Dragomir and Mond proved that the following inequalities of Hermite-Hadamard type hold for log-convex functions:

$$f\Big(\frac{a+b}{2}\Big) \leq \exp\Big\{\frac{1}{b-a}\int_a^b \ln[f(x)]dx\Big\}$$
$$\leq \frac{1}{b-a}\int_a^b G\big(f(x), f(a+b-x)\big)dx$$
$$\leq \frac{1}{b-a}\int_a^b f(x)dx$$
$$\leq L(f(a), f(b)) \leq A(f(a), f(b)). \tag{5}$$

In [15], Pachpatte proved that the following inequalities hold for log-convex functions:

$$\frac{4}{b-a}\int_a^b f(x)g(x)dx \leq \big[f(a)+f(b)\big]L(f(a), f(b))$$
$$+\big[g(a)+g(b)\big]L(g(a), g(b)). \tag{6}$$

In this article, we will establish some Hermite-Hadamard-like type integral inequalities for logarithmically convex functions and some applications for special means are also given.

2 Main results

To reach our goal, we need the following lemmas:

Lemma 1. *[8] For $0 < a < b$ and $0 \le t \le 1$, the following inequality holds:*

$$\sqrt{ab} \ge \begin{cases} a^{1-t}b^t & t \in [0, \frac{1}{2}] \\ a^t b^{1-t} & t \in (\frac{1}{2}, 1], \end{cases} \tag{7}$$

and for $a, b > 0$ and $0 \le t \le 1$, the following inequality holds:

$$2\sqrt{ab} \le a^{1-t}b^t + a^t b^{1-t} \le a + b. \tag{8}$$

Lemma 2. *[26] For $a, b \ge 0$ and $0 \le t \le 1$, the following inequality holds:*

$$(i) \quad G(a,b) = \sqrt{ab} \le \int_0^1 a^t b^{1-t} dt = \int_0^1 a^{1-t} b^t dt \le \frac{a+b}{2} = A(a,b), \tag{9}$$

(ii) *Classical Pólya inequality:*

$$\int_0^1 a^t b^{1-t} dt = L(a,b) \le P(a,b). \tag{10}$$

Lemma 3. *For $x, y \ge 0$, the following inequalities hold:*

$$(i) \quad xy \le x^4 + y^4 + \frac{1}{8}, \tag{11}$$

$$(ii) \quad xy \le \frac{1}{p}x^p + \frac{1}{q}y^q, \quad p > 1, \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{12}$$

Proof. (i) Since $xy \le \frac{1}{2}(x^2 + y^2)$ for $x, y \ge 0$ and the elementary inequality $1 + x^2 \ge 2x$ for $x \ge 0$, we have

$$\begin{aligned} 8(x^4 + y^4) &\ge 4(x^2 + y^2)^2 \\ &\ge (x + y)^4 = 1 + (x + y)^4 - 1 \\ &\ge 2(x + y)^2 - 1 \\ &= 2(x^2 + y^2) + 4xy - 1 \\ &\ge 4xy + 4xy - 1 = 8xy - 1. \end{aligned}$$

(ii) This is Young's inequality.

Theorem 2.1. *Let $f : I \to (0, \infty)$ be a* log-*convex function on an interval I and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(x)dx \le P(f(a), f(b)) \le A(f(a), f(b)). \tag{13}$$

Proof. Since $f : I \to (0, \infty)$ be a log-convex function on an interval I, we have

$$f(ta + (1-t)b) \le [f(a)]^t [f(b)]^{1-t}, \tag{14}$$
$$f((1-t)a + tb) \le [f(a)]^{1-t} [f(b)]^t. \tag{15}$$

By (14) and (15), we have

$$f(ta + (1-t)b) + f((1-t)a + tb)$$
$$\le [f(a)]^t [f(b)]^{1-t} + [f(a)]^{1-t} [f(b)]^t. \tag{16}$$

Integrating both sides of the above inequality (16) over $[0, 1]$, by using (8) we obtain

$$\begin{aligned} &\frac{2}{b-a} \int_a^b f(x)dx \\ &= \int_0^1 \big\{f(ta + (1-t)b) + f((1-t)a + tb)\big\}dt \\ &\le \int_0^1 \big\{[f(a)]^t [f(b)]^{1-t} + [f(a)]^{1-t} [f(b)]^t\big\}dt \\ &= P(f(a), f(b)), \end{aligned}$$

which implies that

$$\frac{1}{b-a} \int_a^b f(x)dx \le L(f(a), f(b)) \le \frac{f(a) + f(b)}{2} = A(f(a), f(b)),$$

where we have used the fact that

$$\begin{aligned} &(i) \int_0^1 \big\{f(ta + (1-t)b) = \frac{1}{b-a} \int_a^b f(x)dx, \\ &(ii) \int_0^1 \big\{f((1-t)a + tb) = \frac{1}{b-a} \int_a^b f(x)dx, \\ &(iii) \int_0^1 [f(a)]^{1-t} [f(b)]^t = \int_0^1 [f(a)]^t [f(b)]^{1-t} dt \le P(f(a), f(b)) \end{aligned}$$

Theorem 2.2. *Let $f : I \to (0, \infty)$ be a* log-*convex function on an interval I and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$\frac{1}{b-a} \int_a^b f(x)f(a+b-x)dx \le G^2(f(a), f(b)).$$

Proof. Since $f : I \to (0, \infty)$ be a log-convex function on an interval I, we have

$$f(ta + (1-t)b) \leq [f(a)]^t [f(b)]^{1-t}, \tag{17}$$
$$f((1-t)a + tb) \leq [f(a)]^{1-t} [f(b)]^t. \tag{18}$$

By (17) and (18), we have

$$(i) \quad \frac{1}{b-a} \int_a^b f(x) f(a+b-x) dx$$
$$\leq \int_0^1 f(ta + (1-t)b) f((1-t)a + tb) dt$$
$$\leq \int_0^1 [f(a)]^t [f(b)]^{1-t} [f(a)]^{1-t} [f(b)]^t dt$$
$$= \int_0^1 f(a) f(b) dt = f(a) f(b) = G^2(f(a), f(b)), \tag{19}$$

$$(ii) \quad \frac{1}{b-a} \int_a^b f(x) f(a+b-x) dx$$
$$\leq \int_0^1 f(ta + (1-t)b) f((1-t)a + tb) dt$$
$$\leq \frac{1}{2} \int_0^1 \Big\{ f^2(ta + (1-t)b) + f^2((1-t)a + tb) \Big\} dt$$
$$\leq \frac{1}{2} \Big[\int_0^1 \big([f(a)]^t [f(b)]^{1-t} \big)^2 dt + \int_0^1 \big([f(a)]^{1-t} [f(b)]^t \big)^2 dt \Big]$$
$$= L\big(f^2(a), f^2(b)\big). \tag{20}$$

Note that

$$G^2(f(a), f(b)) \leq L(f^2(a), f^2(b)) = \frac{f^2(b) - f^2(a)}{2(\ln f(b) - \ln[f(a))}. \tag{21}$$

By (19)-(21), we get the desired result.

Theorem 2.3. *Let $f : I \to (0, \infty)$ be a* log-*convex function on an interval I and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq P(f(a), f(b)). \tag{22}$$

Proof. Since $f : I \to (0, \infty)$ be a log-convex function on an interval I, we have

$$
\begin{aligned}
& f\Big(\frac{a+b}{2}\Big) \\
&= \frac{1}{b-a}\int_a^b f\Big(\frac{a+b-x+x}{2}\Big)dx \\
&\le \frac{1}{b-a}\int_a^b f^{\frac{1}{2}}(x) f(a+b-x) f^{\frac{1}{2}}(x)dx \\
&\le \frac{1}{b-a}\Big\{\int_a^b f(a+b-x)dx\Big\}^{\frac{1}{2}}\Big\{\int_a^b f(x)dx\Big\}^{\frac{1}{2}} \\
&= \frac{1}{b-a}\int_a^b f(x)dx \\
&= \frac{1}{2}\Big[\int_0^1 f(ta+(1-t)b)dt + \int_0^1 f((1-t)a+tb)dt\Big] \\
&\le \frac{1}{2}\Big[\int_0^1 [f(a)]^t[f(b)]^{1-t}dt + \int_0^1 [f(a)]^{1-t}[f(b)]^t dt\Big].
\end{aligned} \tag{23}
$$

By the Pólya inequality (10) and the inequality (23), we have

$$
\begin{aligned}
& f\Big(\frac{a+b}{2}\Big) \\
&\le \frac{1}{b-a}\int_a^b f(x)dx \\
&= \frac{1}{2}\Big[\int_0^1 f(ta+(1-t)b)dt + \int_0^1 f((1-t)a+tb)dt\Big] \\
&\le P(f(a), f(b)) \\
&\le A(f(a), f(b))
\end{aligned}
$$

which get the desired result (22).

Theorem 2.4. *Let $f : I \to (0, \infty)$ be a* log-*convex function on an interval I and $a, b \in I$ with $a < b$. If $g : I \to (0, \infty)$ is positive integrable and symmetric to $x = \frac{a+b}{2}$, then the following inequality holds:*

$$
\begin{aligned}
f\Big(\frac{a+b}{2}\Big)\Big\{\int_a^b \sqrt{g(x)}dx\Big\}^2 &\le \Big\{\int_a^b \sqrt{f(x)g(x)}dx\Big\}^2 \\
&\le (b-a)P(f(a), f(b))\int_a^b g(x)dx.
\end{aligned}
$$

Proof. Since $f : I \to (0, \infty)$ be a log-convex function on an interval I, by Hölder's inequality we have

$$\begin{aligned}
&\sqrt{f\big(\frac{a+b}{2}\big)}\int_a^b \sqrt{g(x)}dx \\
&= \int_a^b \sqrt{f\Big(\frac{a+b-x+x}{2}\Big)}\sqrt{g(x)}dx \\
&\le \int_a^b \sqrt{f^{\frac{1}{2}} f(a+b-x) f^{\frac{1}{2}}(x)}\sqrt{g(x)}dx \\
&= \int_a^b \sqrt{f^{\frac{1}{2}}(a+b-x) g^{\frac{1}{2}}(a+b-x)}\sqrt{f^{\frac{1}{2}}(x)}\sqrt{g^{\frac{1}{2}}(x)}dx \\
&\le \Big\{\int_a^b \sqrt{f(a+b-x)g(a+b-x)}dx\Big\}^{\frac{1}{2}}\Big\{\int_a^b f(x)g(x)dx\Big\}^{\frac{1}{2}} \\
&= \int_a^b \sqrt{f(x)g(x)}dx \\
&\le \Big\{\int_a^b f(x)g(x)dx\Big\}^{\frac{1}{2}}\Big\{\int_a^b dx\Big\}^{\frac{1}{2}} \\
&= (b-a)^{\frac{1}{2}}\Big(\int_0^1 f(x)g(x)dx\Big)^{\frac{1}{2}},
\end{aligned}$$

which implies that

$$\begin{aligned}
&f\big(\frac{a+b}{2}\big)\Big(\int_a^b \sqrt{g(x)}dx\Big)^2 \\
&\le \Big\{\int_a^b \sqrt{f(x)g(x)}dx\Big\}^2 \le (b-a)\int_a^b f(x)g(x)dx. \qquad (24)
\end{aligned}$$

(i) If $0 \le t \le \frac{1}{2}$, then by the inequality (7),(9) and (10) we have

$$\begin{aligned}
&\int_a^b f(x)g(x)dx \\
&= (b-a)\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)dt \\
&\le (b-a)\int_0^1 [f(a)]^t[f(b)]^{1-t} g(ta+(1-t)b)dt \\
&\le \sqrt{[f(a)][f(b)]}\int_a^b g(x)dx \quad \text{(by (7))} \\
&\le \int_0^1 [f(a)]^t[f(b)]^{1-t}dt \int_a^b g(x)dx \quad \text{(by (9))}
\end{aligned}$$

$$\leq P(f(a), f(b)) \int_a^b g(x)dx. \tag{25}$$

(ii) If $\frac{1}{2} \leq t \leq 1$, then by the inequality (7) we have

$$\int_a^b f(x)g(x)dx$$

$$= (b-a) \int_0^1 f((1-t)a + tb)g((1-t)a + tb)dt$$

$$\leq (b-a) \int_0^1 [f(a)]^{1-t}[f(b)]^t g((1-t)a + tb)dt$$

$$\leq \sqrt{[f(a)][f(b)]} \int_a^b g(x)dx \quad \text{(by (7))}$$

$$\leq \int_0^1 [f(a)]^{1-t}[f(b)]^t dt \int_a^b g(x)dx \quad \text{(by (9))}$$

$$\leq P(f(a), f(b)) \int_a^b g(x)dx. \tag{26}$$

By substituting (25) and (26) in (24), we get

$$\begin{aligned}
&f\Big(\frac{a+b}{2}\Big)\Big(\int_a^b \sqrt{g(x)}dx\Big)^2 \\
&\leq \Big\{\int_a^b \sqrt{f(x)g(x)}dx\Big\}^2 \\
&\leq (b-a)^2 \int_0^1 f(ta + (1-t)b)g(ta + (1-t)b)dt \\
&\leq (b-a)^2 \int_0^1 [f(a)]^t[f(b)]^{1-t} g(ta + (1-t)b)dt \\
&\leq (b-a)\sqrt{[f(a)][f(b)]} \int_a^b g(x)dx \\
&\leq (b-a)P(f(a), f(b)) \int_a^b g(x)dx,
\end{aligned}$$

which completes the proof.

Theorem 2.5. *Let $f : I \to (0, \infty)$ be an increasing* log-*convex function on an interval I and $a, b \in I$ with $a < b$. then the following inequality holds:*

$$\begin{aligned}
&f\big(\frac{a+b}{2}\big)L(f(a), f(b)) \\
&\le \int_0^1 f(ta + (1-t)b)[f(a)]^t[f(b)]^{1-t}dt \\
&\le \frac{1}{2}\Big[A(f^2(a), f^2(b)) + L(f^2(a), f^2(b))\Big].
\end{aligned}$$

Proof. Since $f : I \to (0, \infty)$ be an increasing log-convex function on an interval I, we have

$$f(ta + (1-t)b) \le [f(a)]^t[f(b)]^{1-t} \tag{27}$$

for all $a, b \in I$ and $t \in [0, 1]$. Using the inequality $2xy \le x^2 + y^2 (x, y \in R)$, by (27) we have

$$\begin{aligned}
&2f(ta + (1-t)b)[f(a)]^t[f(b)]^{1-t} \\
&\le f^2(ta + (1-t)b) + [f(a)]^{2t}[f(b)]^{2(1-t)}.
\end{aligned} \tag{28}$$

Integrating both sides of the above inequality (28) over $[0, 1]$, we obtain

$$\begin{aligned}
&(i) \quad \int_0^1 f(ta + (1-t)b)[f(a)]^t[f(b)]^{1-t}dt \\
&\le \frac{1}{2}\Big[\int_0^1 f^2(ta + (1-t)b)dt + \int_0^1 [f(a)]^{2t}[f(b)]^{2(1-t)}dt\Big] \\
&= \frac{1}{2}\Big[\frac{1}{b-a}\int_a^b f^2(x)dx + L(f^2(a), f^2(b))\Big] \\
&\le \frac{1}{2}\Big[A(f^2(a), f^2(b)) + L(f^2(a), f^2(b))\Big],
\end{aligned} \tag{29}$$

$$\begin{aligned}
&(ii) \quad \int_0^1 f(ta + (1-t)b)[f(a)]^t[f(b)]^{1-t}dt \\
&\ge \Big\{\int_0^1 f^2(ta + (1-t)b)dt\Big\}\Big\{\int_0^1 [f(a)]^t[f(b)]^{(1-t)}dt\Big\} \\
&= L(f(a), f(b))\Big\{\frac{1}{b-a}\int_a^b f(x)dx\Big\} \\
&\ge f\big(\frac{a+b}{2}\big)L(f(a), f(b)).
\end{aligned} \tag{30}$$

By (29) and (30), we get

$$\begin{aligned}
& f\Big(\frac{a+b}{2}\Big)L(f(a),f(b)) \\
& \le \Big\{\int_0^1 f^2(ta+(1-t)b)dt\Big\}\Big\{\int_0^1 [f(a)]^t[f(b)]^{(1-t)}dt\Big\} \\
& \le \int_0^1 f(ta+(1-t)b)[f(a)]^t[f(b)]^{1-t}dt \\
& \le \frac{1}{2}\Big[\int_0^1 f^2(ta+(1-t)b)dt + \int_0^1 [f(a)]^{2t}[f(b)]^{2(1-t)}dt\Big] \\
& \le \frac{1}{2}\Big[A(f^2(a),f^2(b)) + L(f^2(a),f^2(b))\Big],
\end{aligned}$$

which completes the proof.

Theorem 2.6. *Let $f : I \to (0,\infty)$ be an increasing* log-*convex function on an interval I and $a, b \in I$ with $a < b$. then the following inequality holds:*

$$\begin{aligned}
& f\Big(\frac{a+b}{2}\Big)L(f(a),f(b)) \\
& \le \int_0^1 f(ta+(1-t)b)[f(a)]^t[f(b)]^{1-t}dt \\
& \le \frac{1}{b-a}\int_a^b f^4(x)dx + L(f^4(a),f^4(b)) + \frac{1}{8}.
\end{aligned}$$

Proof. Since $f : I \to (0,\infty)$ be an increasing log-convex function on an interval I, we have

$$f(ta+(1-t)b) \le [f(a)]^t[f(b)]^{1-t} \tag{31}$$

for all $a, b \in I$ and $t \in [0,1]$. Using the inequality $xy \le x^4 + y^4 + \frac{1}{8}(x, y \in R)$ in Lemma 3, by (31) we have

$$\begin{aligned}
& f(ta+(1-t)b)[f(a)]^t[f(b)]^{1-t} \\
& \le f^4(ta+(1-t)b) + [f(a)]^{4t}[f(b)]^{4(1-t)} + \frac{1}{8}.
\end{aligned} \tag{32}$$

Integrating both sides of the above inequality (32) over $[0,1]$, we obtain

$$\int_0^1 f(ta+(1-t)b)[f(a)]^t[f(b)]^{1-t}dt$$

$$\le \int_0^1 f^4(ta+(1-t)b)dt + \int_0^1 [f(a)]^{4t}[f(b)]^{4(1-t)}dt + \frac{1}{8}$$

$$= \frac{1}{b-a}\int_0^1 f^4(x)dt$$

$$+A(f^2(a), f^2(b))A(f(a), f(b))L(f(a), f(b)) + \frac{1}{8} \tag{33}$$

and

$$\int_0^1 f(ta+(1-t)b)[f(a)]^t[f(b)]^{1-t}dt$$

$$\geq \Big\{\int_0^1 f(ta+(1-t)b)dt\Big\}\{\int_0^1 [f(a)]^t[f(b)]^{(1-t)}dt\Big]$$

$$= \Big\{\frac{1}{b-a}\int_a^b f(x)dx\Big\}L(f(a), f(b))$$

$$\geq f\Big(\frac{a+b}{2}\Big)L(f(a), f(b)). \tag{34}$$

By (33) and (34), we get

$$\begin{aligned}
& f\Big(\frac{a+b}{2}\Big)L(f(a), f(b)) \\
& \leq \Big\{\int_0^1 f(ta+(1-t)b)dt\Big\}\{\int_0^1 [f(a)]^t[f(b)]^{(1-t)}dt\Big] \\
& \leq \int_0^1 f(ta+(1-t)b)[f(a)]^t[f(b)]^{1-t}dt \\
& \leq \frac{1}{2}\Big[\int_0^1 f^4(ta+(1-t)b)dt + \int_0^1 [f(a)]^{4t}[f(b)]^{4(1-t)}dt + \frac{1}{8}\Big] \\
& \leq \frac{1}{2}\Big[A(f^4(a), f^4(b)) + L(f^4(a), f^4(b)) + \frac{1}{8}\Big],
\end{aligned}$$

which completes the proof.

Theorem 2.7. *Let $f : I \to (0, \infty)$ be an increasing* log*-convex function on an interval I and $a, b \in I$ with $a < b$. then the following inequality*

$$f\Big(\frac{a+b}{2}\Big)L(f(a), f(b))$$

$$\leq \int_0^1 f(ta+(1-t)b)[f(a)]^t[f(b)]^{1-t}dt$$

$$\leq \frac{1}{q(b-a)}\int_0^1 f^q(x)dx + \frac{1}{p}L(f^p(a), f^p(b)) \tag{35}$$

holds, for $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Since $f : I \to (0, \infty)$ be an increasing log-convex function on an interval I, we have

$$f(ta + (1-t)b) \leq [f(a)]^t [f(b)]^{1-t} \tag{36}$$

for all $a, b \in I$ and $t \in [0, 1]$. Using the inequality $xy \leq \frac{1}{q}x^q + \frac{1}{p}y^p$ for $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ in Lemma 3, by (36) we have

$$f(ta + (1-t)b)[f(a)]^t [f(b)]^{1-t}$$

$$\leq \frac{1}{q} f^q(ta + (1-t)b) + \frac{1}{p}[f(a)]^{pt}[f(b)]^{p(1-t)} \tag{37}$$

for $\frac{1}{p} + \frac{1}{q} = 1$.

Integrating both sides of the above inequality (37) over $[0, 1]$, we obtain

$$\int_0^1 f(ta + (1-t)b)[f(a)]^t [f(b)]^{1-t} dt$$

$$\leq \frac{1}{q} \int_0^1 f^q(ta + (1-t)b)dt + \frac{1}{p} \int_0^1 [f(a)]^{pt}[f(b)]^{p(1-t)} dt$$

$$= \frac{1}{q(b-a)} \int_0^1 f^q(x)dt + \frac{1}{p} L(f^p(a), f^p(b)) \tag{38}$$

and

$$\int_0^1 f(ta + (1-t)b)[f(a)]^t [f(b)]^{1-t} dt$$

$$\geq \Big\{ \int_0^1 f(ta + (1-t)b)dt \Big\} \{ \int_0^1 [f(a)]^t [f(b)]^{(1-t)} dt \Big]$$

$$= \Big\{ \frac{1}{b-a} \int_a^b f(x)dx \Big\} L(f(a), f(b))$$

$$\geq f\Big(\frac{a+b}{2}\Big) L(f(a), f(b)). \tag{39}$$

By (38) and (39), we get

$$\begin{aligned}
&f\big(\frac{a+b}{2}\big)L(f(a),f(b))\\
&\le \Big\{\int_0^1 f(ta+(1-t)b)dt\Big\}\big\{\int_0^1 [f(a)]^t[f(b)]^{(1-t)}dt\Big]\\
&\le \int_0^1 f(ta+(1-t)b)[f(a)]^t[f(b)]^{1-t}dt\\
&\le \frac{1}{q}\int_0^1 f^q(ta+(1-t)b)dt+\frac{1}{p}\int_0^1 [f(a)]^{pt}[f(b)]^{p(1-t)}dt\\
&= \frac{1}{q(b-a)}\int_0^1 f^q(x)dx+\frac{1}{p}L(f^p(a),f^p(b)).
\end{aligned}$$

we get the desired result (35).

3 Applications

The function $f(x)=\frac{1}{x}, x\in(0,\infty)$ is log-convex on $(0,\infty)$. Then we have

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(x)dx=\frac{1}{b-a}\int_a^b \frac{1}{x}dx=L^{-1}(a,b),\\
&f\big(\frac{a+b}{2}\big)=A^{-1}(a,b),\quad \frac{f(a)+f(b)}{2}=H^{-1}(a,b).
\end{aligned}$$

(a) Applying Theorem 2.1 for this function $f(x)=\frac{1}{x}, x\in(0,\infty)$, we get

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b \frac{1}{x}dx=L^{-1}(a,b)\\
&\le \frac{1}{3}\Big\{2G(f(a),f(b))+A(f(a),f(b))\Big\}\\
&= \frac{1}{3}\Big\{\frac{2}{G(a,b)}+\frac{1}{H(a,b)}\Big\}
\end{aligned}\tag{40}$$

Rewriting (40), we get

$$3G(a,b)H(a,b)\le 2L(a,b)\big\{H(a,b)+G(a,b)\big\}.$$

(b) Applying Theorem 2.2 for this function $f(x)=\frac{1}{x}, x\in(0,\infty)$, we get

$$\frac{1}{b-a}\int_a^b f(x)f(a+b-x)dx$$

$$= \frac{1}{b-a}\int_a^b \frac{1}{x(a+b-x)}dx$$
$$= A^{-1}(a,b)L^{-1}(a,b)$$
$$\leq G^2(f(a),f(b)) = \frac{1}{ab}$$
$$= G^{-2}(a,b). \tag{41}$$

Rewriting (41), we get

$$G^2(a,b) \leq A(a,b)L(a,b).$$

Similar inequalities may be stated for the log-convex functions $f(x) = x^x$ for $x > 0$, or $f(x) = e^{x^p}, p \geq 1$ for $x > 0$, or $f(x) = e^x + 1$ for $x \in R$, etc. we omit the details.

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