Int. Journal of Math. Analysis, Vol. 7, 2013, no. 55, 2745 - 2752
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ijma.2013.39224

Hyers-Ulam-Rassias Stability of a Quadratic-Additive Type Functional Equation on a Restricted Domain

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Abstract. In this paper, we prove Hyers-Ulam-Rassias stability of a functional equation

$$f(x+y)+f(x-y)-2f(x)-f(y)-f(-y)=0$$

on a restricted domain by using the direct method in the sense of Hyers.

Mathematics Subject Classification: Primary 39B52

Keywords: Hyers-Ulam-Rassias stability, quadratic-aditive type functional equation

1. Introduction

In 1940, S. M. Ulam [9] presented a problem concerning the stability of group homomorphisms.

The Ulam's problem for the case of approximately additive mappings was solved by Hyers [2]. Indeed, Hyers proved that each solution of the inequality

$$\|f(x+y)-f(x)-f(y)\| \le \varepsilon$$

for all $x \in G_1$ and $y \in G_2$ under the assumption that G_1 and G_2 are Banach spaces, can be approximated by an exact solution of the Cauchy additive functional equation

$$f(x+y)=f(x)+f(y). \tag{1.1}$$

In this case, the Cauchy additive functional equation is said to satisfy the Hyers-Ulam stability. Hyers' result was generalized by T. Aoki[1] for additive mappings and by Th. M. Rassias[7] for linear mappings by allowing the Cauchy difference to be controlled by a sum of powers like

$$\|f(x+y)-f(x)-f(y)\| \le \varepsilon(\|x\|^p+\|y\|^p).$$

In this case, the Cauchy additive functional equation is said to satisfy Hyers-Ulam-Rassias stability.

In 1983, the stability theorem for the quadratic functional equation

$$f(x+y)+f(x-y)-2f(x)-2f(y)=0 \tag{1.2}$$

was proved F. Skof [8]. We call a solution of (1.1) *an additive mapping* and a mapping satisfying (1.2) is called *a quadratic mapping.*

If a mapping is represented by the sum of an additive mapping and a quadratic mapping, we call the mapping a quadratic-additive mapping. For a functional equation $Ef=0$ if all of the solutions of $Ef=0$ are quadratic-additive mappings and all of quadratic-additive mappings are the solutions of $Ef=0$, then we call the functional equation $Ef=0$ a quadratic additive type functional equation. Now we consider the functional equation:

$$f(x+y)+f(x-y)-2f(x)-f(y)-f(-y)=0. \tag{1.3}$$

The functional equations (1.3) is a quadratic-additive type functional equation (see Theorem 2.6 in [5]).

In this paper, we will prove Hyers-Ulam-Rassias stability of quadratic-additive type functional equation (1.3)[3, 4, 6].

2. Main results

Throughout this paper, let X be a normed space and Y a Banach space. For an arbitrary fixed $p\in\mathbb{R}$, put $s=$sign$(2-p)$ and $t=$sign$(1-p)$. For a given mapping $f: X\to Y$, we use the following abbreviations

$$\begin{aligned} J_nf(x) :=& \frac{4^{-sn}}{2}\left(f(2^{sn}x)+f(-2^{sn}x)\right)+\frac{2^{-tn}}{2}\left(f(2^{tn}x)-f(-2^{tn}x)\right),\\ Df(x,y) :=& f(x+y)+f(x-y)-2f(x)-f(y)-f(-y)\end{aligned}$$

for all $x,y\in X$.

Lemma 2.1. *Let $d>0$ be given. If $f: X\to Y$ is a mapping such that $Df(x,y)=0$ for all $x,y\in X$ with $\|x\|,\|y\|\ge d$, then f is a quadratic-additive mapping.*

Proof. Choose $\|x\|\ge d$, then

$$f(0)=\frac{1}{2}(Df(2x,2x)+Df(-x,-x)-Df(3x,x)-2Df(2x,x))=0,$$

i.e., $f(0) = 0$. If $\|x\| \geq d$, $\|y\| < d$ and $\|z\| \geq 3d + \|x\|$, then we have $\|z\|, \|y+2z\|, \|y+z\|, \|x-z\|, \|x+z\| \geq d$. Therefore we get

$$\begin{aligned} Df(x,y) =& - Df(x, y+2z) + 2Df(x,z) + Df(x-z, y+z) \\ &+ Df(x+z, y+z) - Df(y+z, z) - Df(-y-z, -z) = 0 \end{aligned}$$

for all $x, y \in X$ with $\|x\| \geq d$ and $\|y\| < d$. By assumption we have

$$Df(x,y) = 0$$

for all $x, y \in X$ with $\|x\| \geq d$. If $\|x\| < d$ and $\|z\| \geq 3d + \|y\|$, then we have $\|z\|, \|x+2z\|, \|x+z\| \geq d$. Therefore we get

$$\begin{aligned} Df(x,y) =& - Df(x+2z, y) + Df(z, y) + Df(x+z, y-z) \\ &+ Df(x+z, y+z) - 2Df(x+z, z) + Df(-z, -y) = 0 \end{aligned}$$

for all $x, y \in X$ with $\|x\| < d$. By assumption we have

$$Df(x,y) = 0$$

for all $x, y \in X$. □

In the following theorem, we can prove the Hyers-Ulam-Rassias stability of the functional equation (1.3).

Theorem 2.2. *Let $d > 0$ and $p < 1$ be given real numbers. If $f : X \to Y$ is a mapping such that*

$$\|Df(x,y)\| \leq \|x\|^p + \|y\|^p \tag{2.1}$$

for all $x, y \in X$ with $\|x\|, \|y\| \geq d$, then there exists a unique quadratic-additive mapping $F : X \to Y$ such that

$$\|f(x) - F(x)\| \leq \frac{2\|x\|^p}{2-2^p} + \frac{(4 \cdot 2^p + 5 + 3^p)d^p}{6} \quad \text{if} \quad 0 \leq p < 1 \tag{2.2}$$

for all x with $\|x\| \geq d$ and $f(x)$ is itself a quadratic-additive mapping if $p < 0$.

Proof. Since

$$\begin{aligned} \|f(0)\| =& \frac{1}{2}\|Df(2x,2x) + Df(-x,-x) - Df(3x,x) - 2Df(2x,x)\| \\ \leq& \frac{1}{2}(4 \cdot 2^p + 5 + 3^p)\|x\|^p \end{aligned}$$

for all $x \in X$ with $\|x\| \geq d$, we have $f(0) = 0$ if $p < 0$ and $\|f(0)\| \leq \frac{1}{2}(4 \cdot 2^p + 5 + 3^p)d^p$ if $p \geq 0$. Therefore we have $\lim_{n\to\infty} J_n f(0) = 0$ for $p < 1$. By the definitions of $J_n f(x)$ and $Df(x,y)$ we get

$$\begin{aligned} J_n f(x) - J_{n+1} f(x) =& - \frac{Df(2^n x, 2^n x) + Df(-2^n x, -2^n x)}{2 \cdot 4^{n+1}} \\ & - \frac{Df(2^n x, 2^n x) - Df(-2^n x, -2^n x)}{2^{n+2}} + \frac{f(0)}{4^{n+1}} \end{aligned} \tag{2.3}$$

for all $x \in X\backslash\{0\}$ and all nonnegative integer n with $\|2^n x\| \geq d$. It follows from (2.1) and (2.3) that

$$\begin{aligned}
&\|J_n f(x) - J_{n+m} f(x)\| \\
&\quad = \sum_{j=n}^{n+m-1} \|J_j f(x) - J_{j+1} f(x)\| \\
&\quad \leq \sum_{j=n}^{n+m-1} \left(\left\| \frac{(2^{j+1}+1)Df(2^j x, 2^j x)}{2\cdot 4^{j+1}} \right\| + \left\| \frac{(2^{j+1}-1)Df(-2^j x, -2^j x)}{2\cdot 4^{j+1}} \right\| \right. \\
&\qquad\qquad \left. + \left\| \frac{f(0)}{4^{j+1}} \right\| \right) \\
&\quad \leq \begin{cases} \sum_{j=n}^{n+m-1} 2^{-j}\|2^j x\|^p + \frac{\|f(0)\|}{3\cdot 4^n} & \text{if } 0 \leq p < 1, \\ \sum_{j=n}^{n+m-1} 2^{-j}\|2^j x\|^p & \text{if } p < 0 \end{cases} \\
&\quad \leq \begin{cases} \frac{2^{np}\|x\|^p}{2^{n-1}(2-2^p)} + \frac{(4\cdot 2^p+5+3^p)d^p}{6\cdot 4^n} & \text{if } 0 \leq p < 1, \\ \frac{2^{np}\|x\|^p}{2^{n-1}(2-2^p)} & \text{if } p < 0 \end{cases}
\end{aligned} \tag{2.4}$$

for all $x \in X\backslash\{0\}$ and all nonnegative integer n with $\|2^n x\| \geq d$. So, it is easy to show that the sequence $\{J_n f(x)\}$ is a Cauchy sequence for all $x \in X\backslash\{0\}$. Since Y is complete and $\lim_{n\to\infty} J_n f(0) = 0$, the sequence $\{J_n f(x)\}$ converges for all $x \in X$. Hence, we can define a mapping $F : X \to Y$ by

$$F(x) := \lim_{n\to\infty} J_n f(x)$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $m \to \infty$ in (2.4), we get the inequality (2.2) for all x with $\|x\| \geq d$ if $0 \leq p < 1$ and

$$\|f(x) - F(x)\| \leq \frac{2\|x\|^p}{2-2^p} \tag{2.5}$$

for all x with $\|x\| \geq d$ if $p < 0$. From the definition of F, we get

$$\begin{aligned}
DF(x,y) &= \lim_{n\to\infty} \frac{4^{-n}}{2}\left(Df(2^n x, 2^n y) + Df(-2^n x, -2^n y)\right) \\
&\quad + \frac{2^{-n}}{2}\left(Df(2^n x, 2^n y) - Df(-2^n x, -2^n y)\right) \\
&\leq \lim_{n\to\infty} (2^{-(2-p)n} + 2^{-n(1-p)})(\|x\|^p + \|y\|^p) = 0
\end{aligned}$$

for all $x, y \in X\backslash\{0\}$ and all nonnegative integer n with $\|2^n x\|, \|2^n y\| \geq d$. By Lemma 2.1, F is a quadratic-additive mapping.

Now, we are going to show that F is unique. Let $F' : X \to Y$ be another quadratic-additive mapping satisfying (2.2). By Lemma 2.1 and (2.3), it is easy to show that $F'(0) = 0$ and $F'(x) = J_n F'(x)$ for all $n \in \mathbb{N}$ and for all $x \in X$. Since F and F' are quadratic-additive, replacing x by $2^n x$ in (2.2), we have

$$\begin{aligned}\|F(x) - F'(x)\| =&\|J_nF(x) - J_nF'(x)\|\\ \le&\frac{4^{-n}}{2}\big(\|(F-f)(2^nx)\| + \|(f-F')(2^nx)\|\\ &+\|(F-f)(-2^nx)\| + \|(f-F')(-2^nx)\|\big)\\ &+\frac{2^{-n}}{2}\big(\|(F-f)(2^nx)\| + \|(F'-f)(2^nx))\|\\ &+\|(F-f)(-2^nx)\| + \|(F'-f)(-2^nx)\|\big)\\ \le&\left(\frac{2\cdot 2^{np}}{2-2^p}\|x\|^p + \frac{(4\cdot 2^p+5+3^p)d^p}{3}\right)\left(\frac{2}{4^n}+\frac{2}{2^n}\right)\end{aligned}$$

for all $x \in X\backslash\{0\}$ and all positive integers n with $\|2^nx\| \ge d$. Taking the limit in the above inequality as $n \to \infty$, we can conclude that $F(x) = F'(x)$ for all $x \in X$. This proves the uniqueness of F. In the case $p < 0$, from (2.5), we obtain the inequalities

$$\begin{aligned}&\|f(x) - F(x)\|\\ &\quad\le\|Df((k+1)x, kx) - DF((k+1)x, kx)\|\\ &\quad\quad+\|F((2k+1)x) - f((2k+1)x)\| + 2\|f((k+1)x - F((k+1)x)\|\\ &\quad\quad+\|f(kx) - F(kx)\| + \|f(-kx) - F(-kx)\|\\ &\quad\le\left((k+1)^p + k^p + \frac{2(2k+1)^p + 4(k+1)^p + 4k^p}{2-2^p}\right)\|x\|^p\end{aligned}$$

for all $x \ne 0$ and all $k > 0$ with $\|kx\| \ge d$. So we conclude that $f(x) = F(x)$ for all $x \ne 0$ by taking the limit in the above inequality as $k \to \infty$. Since $f(0) = 0$, f is itself a quadratic-additive mapping. □

Theorem 2.3. *Let p be a given real number with $1 < p < 2$. If $f : X \to Y$ is a mapping satisfying (2.1) for all $x, y \in X$, then there exists a unique quadratic-additive mapping $F : X \to Y$ such that*

$$\|f(x) - F(x)\| \le \left(\frac{2}{4-2^p} + \frac{2}{2^p-2}\right)\|x\|^p \tag{2.6}$$

for all $x \in X$.

Proof. Since

$$\|f(0)\| =\frac{1}{2}\|Df(0,0)\| \le \|0\|^p = 0,$$

we have $f(0) = 0$. From the definitions of $J_nf(x)$ and $Df(x,y)$, we get

$$\begin{aligned}
&J_nf(x) - J_{n+1}f(x)\\
&\quad= -\frac{Df(2^nx,2^nx)+Df(-2^nx,-2^nx)}{2\cdot 4^{n+1}}\\
&\qquad+2^{n-1}\left(Df\left(\frac{x}{2^{n+1}},\frac{x}{2^{n+1}}\right)-Df\left(\frac{-x}{2^{n+1}},\frac{-x}{2^{n+1}}\right)\right)
\end{aligned}\tag{2.7}$$

for all $x \in X$ and all nonnegative integer n. It follows from (2.1) and (2.7) that

$$\begin{aligned}
\|J_nf(x) - J_{n+m}f(x)\| &= \sum_{j=n}^{n+m-1}\|J_jf(x) - J_{j+1}f(x)\|\\
&\le \sum_{j=n}^{n+m-1}\left(2^{-2j-1}\|2^jx\|^p + 2^{j+1}\|2^{-j-1}x\|^p\right)\\
&\le \frac{2^{np}\|x\|^p}{2^{2n-1}(4-2^p)} + \frac{2^{n+1}\|x\|^p}{2^{np}(2^p-2)}
\end{aligned}\tag{2.8}$$

for all $x \in X$. So, it is easy to show that the sequence $\{J_nf(x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, we can define a mapping $F : X \to Y$ by

$$F(x) := \lim_{n\to\infty} J_nf(x)$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $m \to \infty$ in (2.8), we get the inequality (2.6). From the definition of F, we get

$$\begin{aligned}
DF(x,y) &= \lim_{n\to\infty}\frac{4^{-n}}{2}\left(Df(2^nx,2^ny)+Df(-2^nx,-2^ny)\right)\\
&\quad+\frac{2^n}{2}\left(Df(2^{-n}x,2^{-n}y)-Df(-2^{-n}x,-2^{-n}y)\right)\\
&\le \lim_{n\to\infty}(2^{(p-2)n}+2^{n(1-p)})(\|x\|^p+\|y\|^p) = 0
\end{aligned}$$

for all $x, y \in X$. So, F is a quadratic-additive mapping.

Now, to show that F is unique. Let $F' : X \to Y$ be another quadratic-additive mapping satisfying (2.6). By Lemma 2.1 and (2.7), it is easy to show that $F'(0) = 0$ and $F'(x) = J_nF'(x)$ for all $n \in \mathbb{N}$ and all $x \in X$. From these and (2.6), we have

$$\begin{aligned}\|F(x)-F'(x)\| =&\|J_nF(x)-J_nF'(x)\|\\ \leq&\frac{4^{-n}}{2}\big(\|(F-f)(2^nx)\|+\|(f-F')(2^nx)\|\\ &+\|(F-f)(-2^nx)\|+\|(f-F')(-2^nx)\|\big)\\ &+\frac{2^n}{2}\big(\|(F-f)(2^{-n}x)\|+\|(F'-f)(2^{-n}x))\|\\ &+\|(F-f)(-2^{-n}x)\|+\|(F'-f)(-2^{-n}x)\|\big)\\ \leq&\left(\frac{4}{|4-2^p|}+\frac{4}{|2^p-2|}\right)(2^{(p-2)n}+2^{(1-p)n})\|x\|^p\end{aligned}$$

for all $x\in X$ and all positive integer n. Taking the limit in the above inequality as $n\to\infty$, we can conclude that $F(x)=F'(x)$ for all $x\in X$. This proves the uniqueness of F. □

Theorem 2.4. *Let $d>0$ and p be given real numbers with $p>2$. If $f:X\to Y$ is a mapping satisfying (2.1) for all $x,y\in X$ with $\|x\|,\|y\|\leq d$, then there exists a unique quadratic-additive mapping $F:X\to Y$ such that*

$$\|f(x)-F(x)\|\leq\frac{2\|x\|^p}{2^p-4} \tag{2.9}$$

for all $\|x\|\leq 2d$.

Proof. Since

$$\|f(0)\|=\frac{1}{2}\|Df(0,0)\|\leq\|0\|^p=0,$$

we have $f(0)=0$. From the definitions of $J_nf(x)$ and $Df(x,y)$, we get

$$\begin{aligned}J_nf(x)-J_{n+1}f(x)=&(2^{2n-1}+2^{n-1})Df\left(\frac{x}{2^{n+1}},\frac{x}{2^{n+1}}\right)\\ &+(2^{2n-1}-2^{n-1})Df\left(\frac{-x}{2^{n+1}},\frac{-x}{2^{n+1}}\right)\end{aligned} \tag{2.10}$$

for all $x\in X$ and all nonnegative integer n with $\|2^{-n-1}x\|\leq d$. It follows from (2.1) and (2.10) that

$$\begin{aligned}\|J_nf(x)-J_{n+m}f(x)\|\leq&\sum_{j=n}^{n+m-1}2^{2j+1}\|2^{-j-1}x\|^p\\ \leq&\frac{2^{2n+1}\|x\|^p}{2^{np}(2^p-4)}\end{aligned} \tag{2.11}$$

for all $x\in X$ and all nonnegative integer n with $\|2^{-n-1}x\|\leq d$. So, it is easy to show that the sequence $\{J_nf(x)\}$ is a Cauchy sequence for all $x\in X$. Since Y is complete, we can define a mapping $F:X\to Y$ by

$$F(x):=\lim_{n\to\infty}J_nf(x)$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $m \to \infty$ in (2.11), we get the inequality (2.9). From the definition of F, we get

$$\begin{aligned} DF(x,y) = & \lim_{n\to\infty} 2^{2n-1}\left(Df(2^{-n}x,2^{-n}y)+Df(-2^{-n}x,-2^{-n}y)\right) \\ & + 2^{n-1}\left(Df(2^{-n}x,2^{-n}y)-Df(-2^{-n}x,-2^{-n}y)\right) \\ \le & \lim_{n\to\infty}(2^{(2-p)n}+2^{(1-p)n})(\|x\|^p+\|y\|^p) = 0 \end{aligned}$$

for all $x, y \in X$ and all nonnegative integer n with $\|2^{-n}x\|, \|2^{-n}y\| \le d$. So, F is a quadratic-additive mapping.

Now, to show that F is unique. Let $F' : X \to Y$ be another quadratic-additive mapping satisfying (2.9). By Lemma 2.1 and (2.10), it is easy to show that $F'(0) = 0$ and $F'(x) = J_nF'(x)$ for all $n \in \mathbb{N}$ and for all $x \in X$. Since F and F' are quadratic-additive, we have

$$\begin{aligned} \|F(x)-F'(x)\| = & \|J_nF(x)-J_nF'(x)\| \\ \le & 2^{2n-1}\big(\|(F-f)(2^{-n}x)\|+\|(f-F')(2^{-n}x)\| \\ & +\|(F-f)(-2^{-n}x)\|+\|(f-F')(-2^{-n}x)\|\big) \\ & + 2^{n-1}\big(\|(F-f)(2^{-n}x)\|+\|(F'-f)(2^{-n}x))\| \\ & +\|(F-f)(-2^{-n}x)\|+\|(F'-f)(-2^{-n}x)\|\big) \\ \le & \left(\frac{4\|x\|^p}{2^p-4}\right)(2^{(2-p)n}+2^{(1-p)n})\|x\|^p \end{aligned}$$

for all $x \in X$ and all positive integer n with $\|2^{-n}x\| \le d$. Taking the limit in the above inequality as $n \to \infty$, we can conclude that $F(x) = F'(x)$ for all $x \in X$. This proves the uniqueness of F. □

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Received: September 25, 2013