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Cyclic Contractions and Fixed Points in Dislocated Metric Spaces

Reny George[1]

Department of Mathematics, College of Science
Salman Bin Abdulaziz University, Al Kharj, Saudi Arabia
renygeorge02@yahoo.com

R. Rajagopalan[2]

Department of Mathematics, College of Science
Salman Bin Abdulaziz University, Al Kharj, Saudi Arabia

S. Vinayagam

Department of Mathematics, Sree Saraswathi Thyagaraja College
Pollachi, Coimbatore, Tamilnadu, India

Abstract. Fixed point theorems for various cyclic contractions are proved in dislocated metric space. Our result generalises many well known fixed point theorems.

Keywords: Dislocated metric space, fixed points, cyclic mappings, d-cyclic contractions

1. Introduction

In 2003 Kirk et al.[2] introduced cyclic contractions in metric spaces and investigated the existence of proximity points and fixed points for cyclic contraction mappings. Since then many results appeared in this field. (see [3],[4],[6],[7],[8],[9],[10]). In 2009, Sankar Raj and Veeramani [11] proved exisitence of best proximity points for relatively non expansive maps. Later, Karapinar et al.[4] proved the existence of fixed points for various types of

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[1]Permanent affiliation: Department of Mathematics and Computer Science, St. Thomas College, Bhilai, CG, India

[2]Permanent affiliation:Department of Mathematics, Singhania University, Jhunjhunu, Rajasthan, India

cyclic contractions in metric space. The purpose of this paper is to prove fixed point results for various cyclic contractions in dislocated metric spaces which generalises many known results.

2. Preliminaries

Definition 2.1 Let X be a non-empty set. The distance function $d : X \times X \to [0, \infty)$ is a metric, if it satisfies the following:

(1) $d(x, x) = 0$

(2) $d(x, y) \geq 0$

(3) $d(x, y) = 0 \Rightarrow x = y$

(4) $d(x, y) = d(y, x)$

(5) $d(x, y) \leq d(x, z) + d(z, y)$

If d satisfies (2) to (5) above, then d is called a dislocated metric and (X, d) is a dislocated metric space(d-metric space). Clearly every metric space is a dislocated metric space but the converse is not necessarily true. (see [5])

Definition 2.2[5] A sequence $\{x_n\}$ in a d-metric space is a cauchy sequence if for all $\epsilon > 0$, there exists N_0 such that for all $m, n \geq N_0$, $d(x_m, x_n) \leq \epsilon$

Definition 2.3[5] A sequence $\{x_n\}$ in a d-metric d-converges to $x \in X$ if for $\epsilon > 0$, there exists N_0 such that for all $n \geq N_0$, $lim_{n\to\infty} d(x_n, x) < \epsilon$

Definition 2.4[5] The d-metric space (X, d) is d-complete if every cauchy sequence in it is d-convergent.

Definition 2.5 [7] Let A and B be non empty subsets of a metric space (X, d) and $T : A \bigcup B \to A \bigcup B$. T is called a cyclic map iff $T(A) \in B$ and $T(B) \in A$.

Definition 2.6[7] Let A and B be non-empty subsets of a metric space (X, d). A cyclic map $T : A \bigcup B \to A \bigcup B$ is said to be a cyclic contraction if there exists $k \in (0, 1)$ such that $d(Tx, Ty) \leq k(d(x, y))$ for all $x \in A$ and $y \in B$.

Definition 2.7[4] Let A and B be non empty closed subsets of a metric space (X, d). A cyclic mapping $T : A \bigcup B \to A \bigcup B$ is called a Kannan type cyclic contraction if there exist $k \in (0, 1/2)$ such that $d(Tx, Ty) \leq k[d(Tx, x) + d(Ty, y)]$ for all $x \in A$ and $y \in B$.

In Karapinar et al [4], it has been shown that Kannan type cyclic contraction and cyclic contraction are independent of each other.

Definition 2.8[4] Let A and B be non empty closed subsets of a metric space (X, d). A cyclic mapping $T : A \bigcup B \to A \bigcup B$ is called a Chatterjee type cyclic contraction if there exist $k \in (0, 1/2)$ such that $d(Tx, Ty) \leq k[d(Tx, y) + d(Ty, x)]$ for all $x \in A$ and $y \in B$.

Definition 2.9[4] Let A and B be non empty closed subsets of a metric space (X, d). A cyclic mapping $T : A \bigcup B \to A \bigcup B$ is called a Ciric type cyclic contraction if there exist $k \in (0, 1)$ such that $d(Tx, Ty) \leq kMax\{d(x, y), d(Tx, x), d(Ty, y)\}$ for all $x \in A$ and $y \in B$.

3. MAIN RESULTS:

Definition 3.1 Let A and B be non empty closed subsets of a dislocated metric space (X, d). We say that, a cyclic mapping $T : A \bigcup B \to A \bigcup B$ is a d-cyclic contraction if there exist $k \in (0, 1/2)$ such that $d(Tx, Ty) \leq kd(x, y)$ for all $x \in A$ and $y \in B$.

Example 3.2 Let $X = R$, $d(x, y) = \mid x - y \mid + 4 \mid x \mid + 4 \mid y \mid$. Then (X, d) is a dislocated metric space but not a metric space. Let $A = B = [0, 1]$ and define $T : A \bigcup B \to A \bigcup B$ by

$$Tx = \begin{cases} 1/4, x = 1 \\ 1, x \in [0, 1) \end{cases}$$

We see that T is a d-cyclic contraction in the d-metric space. However for $x = \frac{15}{16}$ and $y = 1$, the cyclic contraction condition fails in usual metric space.

Theorem 3.3 Let (X, d) be a d-metric space, A and B be non empty closed subsets of X and $T : A \bigcup B \to A \bigcup B$ be a d-cyclic contraction in X. Then T has a unique fixed point in $A \bigcap B$.

Proof. Fix $x \in A$. By definition 3.1, there exist $k \in (0, 1/2)$ such that $d(T^2x, Tx) \leq kd(Tx, x)$. Inductively, we have,

$$d(T^{n+1}x, T^nx) \leq k^nd(Tx, x)$$

More generally, for $m > n$, we have

$d(T^m x, T^n x) \leq d(T^m x, T^{m-1} x) + d(T^{m-1} x + T^{m-2} x) + ...d(T^{n+1} x, T^n x)$

$\leq [k^{m-1} + k^{m-2} + ...k^n] d(Tx, x)$

$\leq k^n [1 + k + k^2 + k^3 + ... + k^{m-n-1}] d(Tx, x)$

Since $k \in (0, 1/2)$, as $m \to \infty$ and $n \to \infty$, we have

$k^n [1 + k + k^2 + k^3 + ... + k^{m-n-1}]) \to 0 \Rightarrow d(T^m x, T^n x) \to 0$. Hence $< T^n x >$ is a cauchy sequence. Since (X, d) is complete, we have, $< T^n x >$ converges to some $z \in X$. Note that $< T^{2n} x >$ is a sequence in A, $< T^{2n-1} x >$ is a sequence in B and so $z \in A \bigcap B$.

We claim $Tz = z$. We have

$d(Tz, z) = d(Tz, lim_{n \to \infty} T^{2n} x)$

$\leq k.d(z, lim_{n \to \infty} T^{2n-1} x)$

$\leq k.d(z, z) \leq k[d(Tz, z) + d(z, Tz)] = 2kd(Tz, z)$

i.e. $(1 - 2k) d(Tz, z) \leq 0 \Rightarrow d(Tz, z) = 0$

$\Rightarrow Tz = z$. Thus z is a fixed point of T.

To show the uniqueness, let us assume that there exist two fixed points say u and v such that $Tu = u$ and $Tv = v$.

Now, $d(Tu, Tv) \leq kd(u, v)$

i.e.,$d(u, v) \leq kd(u, v)$

$(1 - k) d(u, v) \leq 0$

Since $k \in (0, 1/2)$, we have $d(u, v) = 0 \Rightarrow$ u=v.

□

Definition 3.4 Let A and B be non empty closed subsets of a dislocated metric space (X, d). We say that, a cyclic mapping $T : A \bigcup B \to A \bigcup B$ is a Kannan type d-cyclic contraction if there exist $k \in (0, 1/3)$ such that

$d(Tx, Ty) \leq k[d(Tx, x) + d(Ty, y)]$ for all $x \in A$ and $y \in B$.

Example 3.5 Let $X = R$, $d(x, y) = | x - y | + 4| x | + 4| y |.$, $A = [-1, 0]$ and $B = [0, 1]$. Define $T : A \bigcup B \to A \bigcup B$ by $Tx = \frac{-x}{3}$. T is a Kannan type d-cyclic contraction in the d-metric (X, d). However, T is not a Kannan type cyclic contraction in usual metric space.

Theorem 3.6 Let (X, d) be a d-metric space, A and B be non empty closed subsets of X and $T : A \bigcup B \to A \bigcup B$ be a Kannan type d-cyclic contraction in X. Then T has a unique fixed point in $A \bigcap B$.

Proof. Fix $x \in A$. By definition 3.4, there exist $k \in (0, 1/3)$ such that

$d(T^2x, Tx) \leq k[d(T^2x, Tx) + d(Tx, x)]$

$\Rightarrow (1 - k)d(T^2x, Tx) \leq k[d(Tx, x)]$

$d(T^2x, Tx) \leq \frac{k}{1-k}[d(Tx, x)] = t[d(Tx, x)]$, where $t = \frac{k}{1-k} \in (0, 1/2)$.

Inductively, we have,

$d(T^{n+1}x, T^nx) \leq t^n d(Tx, x)$.

Hence proceeding as in Theorem 3.3, we see that $< T^nx >$ is a Cauchy sequence in X. Since (X, d) is complete, we have, $< T^nx >$ converges to some $z \in X$. Note that $< T^{2n}x >$ is a sequence in A, $< T^{2n-1}x >$ is a sequence in B and so $z \in A \bigcap B$.

We claim $Tz = z$.

Now, $d(Tz, T^{2n}x) \leq k[d(Tz, z) + d(T^{2n}x), T^{2n-1}x)]$. Hence as $n \to \infty$ we have

$d(Tz, z) \leq k[d(Tz, z) + d(z, z)]$

$\leq k[d(Tz, z) + d(z, Tz) + d(Tz, z)]$

$\leq 3kd(Tz, z)$

i.e. $(1 - 3k)d(Tz, z) \leq 0 \Rightarrow d(Tz, z) = 0 \Rightarrow Tz = z$. Thus z is a fixed point.

To show the uniqueness, let us assume that there exists two fixed points say u and v such that $Tu = u$ and $Tv = v$.

Now, $d(Tu,Tv) \leq k[d(Tu,u)+d(Tv,v)]$

$\Rightarrow d(u,v) \leq k[d(u,u)+d(v,v)]$.......(1)

$d(Tu,Tu) \leq k[d(Tu,u)+d(Tu,u)]$

$\Rightarrow d(u,u) \leq k[d(u,u)+d(u,u)]$

$\Rightarrow (1-2k)d(u,u) \leq 0.$

Since $k \in (0,1/3)$, we have $d(u,u)=0$. Similarly we can show that $d(v,v)=0$. Thus (1) becomes, $d(u,v) \leq 0 \Rightarrow u=v$. □

Definition 3.7 Let A and B be non empty closed subsets of a dislocated metric space (X,d). We say that a cyclic mapping $T : A \bigcup B \rightarrow A \bigcup B$ is a Chatterjee type d-cyclic contraction if there exist $k \in (0,1/3)$ such that $d(Tx,Ty) \leq k[d(Tx,y)+d(Ty,x)]$ for all $x \in A$ and $y \in B$.

Theorem 3.8 Let (X,d) be a d-metric space, A and B be non empty closed subsets of X and $T : A \bigcup B \rightarrow A \bigcup B$ be a Chatterjee type d-cyclic contraction in X. Then T has a unique fixed point in $A \bigcap B$.

Proof. Fix $x \in A$. By definition 3.7, there exist $k \in (0,1/3)$ such that

d(T^2x, Tx) $\leq k[d(T^2x,x)+d(Tx,Tx)]$

$\Rightarrow (1-k)d(T^2x,Tx) \leq k[d(Tx,x)+d(x,Tx)]$

$\Rightarrow d(T^2x,Tx) \leq \frac{2k}{1-k}[d(Tx,x)] = t[d(Tx,x)]$, where $t=\frac{2k}{1-k} \in (0,1)$.

Inductively, we have,

$d(T^{n+1}x,T^nx) \leq t^n d(Tx,x)$.

Hence proceeding as in Theorem 3.3, we see that $< T^nx >$ is a Cauchy sequence in X. As (X,d) is complete, we have, $< T^nx >$ converges to some $z \in X$. Note that $< T^{2n}x >$ is a sequence in A, $< T^{2n-1}x >$ is a sequence in B and so $z \in A \bigcap B$.

We claim $Tz = z$.

By definition 3.7,

$d(Tz, T^{2n}x)) \leq k[d(Tz, T^{2n-1}x + d(T^{2n}x, z)]$. Hence as $n \to \infty$ we have

$d(Tz, z) \leq k[d(Tz, z) + d(z, z)]$

$\leq k[d(Tz, z) + d(z, Tz) + d(Tz, z)]$

$= 3kd(Tz, z)$

i.e. $(1 - 3k)d(Tz, z) \leq 0 \Rightarrow d(Tz, z) = 0 \Rightarrow Tz = z$. Thus z is a fixed point of T.

To show the uniqueness, let us assume that there exists two fixed points say u and v such that $Tu = u$ and $Tv = v$.

Now, $d(Tu, Tv) \leq k[d(Tu, v) + d(Tv, u)]$

i.e.,$d(u, v) \leq k[d(u, v) + d(v, u)]$

$\Rightarrow d(u, v) \leq 2k[d(u, v)]$

$\Rightarrow (1 - 2k)d(u, v) \leq 0$

Since $k \in (0, 1/3)$, we have $d(u, v) = 0 \Rightarrow u = v$. □

Definition 3.9 Let A and B be non empty closed subsets of a dislocated metric space (X, d). We say that a cyclic mapping $T : A \bigcup B \to A \bigcup B$ is a Ciric type d-cyclic contraction if there exist $k \in (0, 1/2)$ such that

$d(Tx, Ty) \leq kMax\{d(x, y), d(Tx, x), d(Ty, y)\}$ for all $x \in A$ and $y \in B$.

Theorem 3.10 Let (X, d) be a d-metric space, A and B be non empty closed subsets of X and $T : A \bigcup B \to A \bigcup B$ be a Ciric type d-cyclic contraction in X. Then T has a unique fixed point in $A \bigcap B$.

Proof. Fix $x \in A$. If $T^n x = T^{n+1}x$ for some n, then $T^{n+1}x = T^{n+2}x$ and thus $< T^n x >$ converges to some $z \in X$. Suppose $T^n x \neq T^{n+1}x$. By definition 3.9, there exist $k \in (0, 1/2)$ such that

$d(T^2x, Tx) \leq kMax\{d(T^2x, Tx), d(Tx, x), d(Tx, x)\}$

$\Rightarrow d(T^2x, Tx) \leq kMax\{d(T^2x, Tx), d(Tx, x)\}$

i.e., $d(T^2x, Tx) \leq k[d(Tx, x)]$ (Since $T^2x \neq Tx$). Inductively, we have,

$d(T^{n+1}x, T^n x) \leq k^n[d(Tx, x)$.

Hence proceeding as in Theorem 3.3, we see that $< T^n x >$ is a Cauchy sequence in X. As (X, d) is complete, we have, $< T^n x >$ converges to some $z \in X$. Note that $< T^{2n}x >$ is a sequence in A, $< T^{2n-1}x >$ is a sequence in B and so $z \in A \bigcap B$

We claim $Tz = z$

$d(Tz, T^{2n}x)) \leq kMax\{d(z, T^{2n-1}x), d(Tz, z), d(T^{2n}x, T^{2n-1}x)\}$.

Hence as $n \to \infty$ we have

$d(Tz, z) \leq kMax\{d(z, z), d(Tz, z), d(z, z)\}$

$\leq kd(Tz, z)$ or $kd(z, z)$

$\leq kd(Tz, z)$ or $2kd(Tz, z)$

$\Rightarrow (1 - k)d(Tz, z) \leq 0$ or $(1 - 2k)d(Tz, z) \leq 0 \Rightarrow d(Tz, z) = 0$

$\Rightarrow Tz = z$. Thus z is a fixed point.

To show the uniqueness, let us assume that there exists two fixed points say u and v such that $Tu = u$ and $Tv = v$.

Now, $d(Tu, Tv) \leq kMax\{d(Tu, u), d(Tv, v), d(u, v)\}$

i.e., $d(u, v) \leq kMax\{d(u, u), d(v, v), d(u, v)\}$

$\Rightarrow d(u, v) \leq kd(u, u)$ or $kd(v, v)$ or $kd(u, v)$

$\Rightarrow d(u, v) \leq 2kd(u, v)$ or $kd(u, v)$

$\Rightarrow (1 - 2k)d(u, v) \leq 0$ or $(1 - 2k)d(u, v) \leq 0$.

Since $k \in (0, 1/2)$, we have $d(u, v) = 0$

$\Rightarrow u = v$.

□

References

1. M.A.Al Thafai and N Shahzad (2009) Convergence and Existence for Best Proximity Points, *Nonlinear Analysis*, 70, 3665-3671.

2. A. A Eldered and P Veeramani (2009) Proximal Pointwise Contractions. *Topology and its Applications*, 156, 2942-2948.

3. A. A Eldered and P Veeramani (2006) Convergence and existence for Best proximity Points. *J. Math Analysis and Applications*, 323, 1001-1006.

4. Erdal Karapinar and Inci M Erhan (2010) Best Proximity on Different Type Contractions *Applied Mathematics and Information Science* (in press)

5. P. Hitzler and A. K. Seda (2000) Dislocated topologies, *Proc. of the slovakian conference in applied mathematics* , Bratislava.

6. S. Karpagam and Sushma Agrawal (2010) Best Proximity Points theorems for Cyclic Meir Keeler Contraction Maps, *Nonlinear Analysis* doi:10.1016/j.na.2010.07.26(inpress)

7. W.A.Kirk and P.S. Srinivasan and P.Veeramani(2003) Fixed Points for mapping satsifying Cyclic contractive conditions. *Fixed Point Theory*, 4, 79-89.

8. G Petrushel (2005), Cyclic representations and Periodic points, *Studia Univ. Babes - Bloyai Math*, 50, 107-112.

9. Pacurar M., Rus, I.A - (2010) Fixed Point Theory for ϕ -contractions, *Nonlinear Anlaysis*, 72,(3-4), 1181-1187.

10. Sh.Rezapur and M.Derafshpour and N.Shahzad - Best Proximity point of cyclic ϕ contractions in ordered metric spaces *Topological Methods in Nonlinear Analysis* (in press).

11. V Sankar Raj and P Veeramani (2009) - Best Proximity Pair Theorems for relatively Non Expansive Mappings *Applied General Topology*, 10, (1) 21-28 (in press).

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