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Fuzzy Delay Predator-Prey System: Existence Theorem and Oscillation Property of Solution

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Abstract

This paper discusses the existence of solution for a fuzzy delay predator-prey system. First, we prove the existence theorem of at least one solution of fuzzy predator-prey system where the initial condition is also described by a fuzzy number on E^2 space. The results are obtained by the fixed point principles. We also determine the necessary and sufficient conditions for oscillation of solutions for the system. Finally, some examples are given to explain the existence of the solution and oscillation property.

Mathematics Subject Classification: 74H20

Keywords: Fuzzy variable; Levelwise Continuous function ; Fuzzy Solution; Fuzzy Delay Predator-Prey System (FDPPS); Oscillation solution

1 Introduction

Volterra and Lokta introduced the predator-prey model for the study of population dynamics in 1920. Over the years, variations and extensions were made to this model. In recent years, the studies of such models with relation to stabilities and oscillatory behavior have been popular. In the analysis of population dynamics Predator-prey forms the basis of many models used today, and it is one of the most useful systems applied to mathematical ecology. The dynamic properties of the predator-prey models have been given great attentions which have significant biological background.

In the field of predator-prey interaction, the studies on population were extended by including time delay and harvesting. A time delay is considered when the rate of changes of population is not only a function of the present population but also depends on the past population. In the model with harvesting, some studies relate the population to the economic problems. There exists a great range of literatures on delay predator-prey equations (see, for instance, Ruan [14] and references cited therein). Fuzzy differential equations are a natural way to model dynamical systems under uncertainty. This type of system will provide a better representation of the real world problems. Therefore the study of this topic has been rapidly growing in recent years. Chang and Zadeh [4] were the first, to introduce the concept of fuzzy derivatives. It was followed up by Dubois and Prade [5] in 1982. They used the extension principle in their approach. In 1980 Kandel and Byatt [8] applied the concept of fuzzy differential equation to the analysis of fuzzy dynamical problems. Many researches on the theory of fuzzy differential equations and their application to real problems were reported [3, 15, 10].

The existence and uniqueness theorem is one of the most important and fundamental theorems in the theory of classical differential equation. The theorem in the case of fuzzy-set function were also discussed in many literatures. Some of them can be found in [16, 17, 2, 13]. Xintong [16], introduced uncertain delay differential equations and proved the existence and uniqueness theorem for the equations under Lipschitz and linear growth conditions by Banach fixed point theorem. Xiaowei Chen and Zhongfeng Qin [17] proved a new existence and uniqueness theorem for fuzzy differential equation. This theorem was different from previous works since they use Liu process [17]. Balachandran and Prakash [2] proved the existence of solutions of fuzzy delay differential equations with nonlocal condition. Park [13] proved an existence and uniqueness theorem for fuzzy differential equations by using successive approximation on E^n. This existence theorem deals with a special class of fuzzy differential equation.

In this paper the existence of solution for fuzzy delay predator-prey with fuzzy initial condition on (E^2, D) will be discussed.

Apart from the fundamental existence theorem, an important area of study on delay differential equations deals with the nature of the solutions. In particular, the oscillatory behavior of solutions of such equation is believed to be of great importance. A recent survey on oscillation theory of delay differential equations were found in [6], Ladde [12] and reference therein. However, to our knowledge, only a few results for the oscillatory property of some fuzzy differential system have been established [7].

In this study we shall look at oscillation problem for fuzzy delay predator-prey system, we establish necessary and sufficient conditions for all solutions to be oscillatory.

This paper is organized as follows. In Section 2, some notations, concepts and terminologies are given. In Section 3, the formulation of the fuzzy delay differential predator-prey system are presented . In Section 4, we prove the existence theorem for fuzzy delay predator-prey system. In Section 5, we discuss the oscillation solution of the fuzzy delay predator-prey system. Some examples are presented in Section 6. Finally, Section 7 provides a brief conclusion.

2 Preliminary Notes / Materials and Methods

Consider $P(R^n)$ be the family of all nonempty, compact, convex subsets of R^n. Addition and scalar multiplication are defined in $P(R^n)$. Let A, B be two nonempty bounded subsets of R^n. The distance between A, B can be defined by the Hausdorff metric

$$d(A,B) = max\{\sup_{a\in A}\inf_{b\in B} \parallel a-b \parallel, \sup_{b\in B}\inf_{a\in A} \parallel a-b \parallel\},$$

where $\parallel . \parallel$ is denoted as an usual Euclidean norm in R^n, so that $(P(R^n), d)$ becomes a metric spaces. Assume that $I = [t_0, t_0 + a] \subset R(a > 0)$ is a compact interval and $E^n = \{u : R^n \to [0, 1]\}$ such that u satisfies the following properties:

1. u is normal, that is there exists an $x_0 \in R^n$ such that $u(x_0) = 1$,

2. u is fuzzy convex, that is,
   $u(\lambda x + (1-\lambda)y) \geq min\{u(x), u(y)\}$, for any $x, y \in R^n$ and $0 \leq \lambda \leq 1$,
3. u is upper semicontinuous,
4. $[u]_0 = cl\{x \in R^n : u(x) > 0\}$ is compact.

If $u \in E^n$, then u is called a fuzzy number, and E^n is called a fuzzy number space. For $0 \leq \alpha \leq 1$, denote $[u]_\alpha = \{x \in R^n : u(x) \geq \alpha\}$. From $(1)-(4)$, it follows that the α-level set $[u]_\alpha \in P(R^n)$ for $0 \leq \alpha \leq 1$. Also, we have $[u+v]_\alpha = [u]_\alpha + [v]_\alpha$, $[ku]_\alpha = k[u]_\alpha$, where $k \in R$.

Define D:$E^n \times E^n \to [0, \infty)$ by the relation $D(u, v) = \sup_{0 \leq \alpha \leq 1} d([u]_\alpha, [v]_\alpha)$, where d is the Hausdorf metric defined in $P(R^n)$. Hence D is a metric in E^n.

Further we have that [2]

1. (E^n, D) is a complete metric space,
2. $D(u+w, v+w) = D(u, v)$ for all $u, v, w \in E^n$,
3. $D(\lambda u, \lambda v) = |\lambda| D(u, v)$ for all $u, v \in E^n$ and $\lambda \in R$.

It can be showed that $D(u+v, w+z) \leq D(u, w) + D(v, z)$ for all u, v, w and $z \in E^n$.

Definition 2.1 *[9] The mapping $F : I \to E^n$ is said to be strongly measurable if for all $\alpha \in [0,1]$ the set- valued map $F_\alpha : I \to P(R^n)$ defined by $F_\alpha(t) = [F(t)]_\alpha$ is Lebesgue measurable where $P(R^n)$ has the topology induced by the Hausdorff metric d.*

Definition 2.2 *[9] The mapping $F : I \to E^n$ is said to be integrably bounded if there is an integrable function $h(t)$ such that $\|x(t)\| \leq h(t)$ for every $x(t) \in F_0(t)$.*

Definition 2.3 *[2] The fuzzy mapping $F : I \to E^n$ integral is defined levelwise by $[\int_I F_\alpha(t)dt] = \int_I F_\alpha(t)dt =$ The set of all $\int_I f(t)dt$ such that $f : I \to R^n$ is measurable selection for F_α for all $\alpha \in [0,1]$.*

Definition 2.4 *[1] $F : I \to E^n$ is said to be integrable on I if F is a strongly measurable and integrably bounded mapping such that$\int_I F(t)dt \in E^n$.*

Any strongly measurable and integrably bounded function $F : I \to E^n$ is integrable. Further more if F is continuous, then it is also integrable.

Proposition 2.5 *[2] If $F : I \to E^n$ and G:$I \to E^n$ are integrable and $c \in I, \lambda \in R$. Then*

1. $\int_{t_0}^{t_0+a} F(t)dt = \int_{t_0}^{c} F(t)dt + \int_{c}^{t_0+a} F(t)dt$,
2. $\int_I (F(t) + G(t))dt = \int_I F(t)dt + \int_I G(t)dt$,
3. $\int_I \lambda F(t)dt = \lambda \int_I F(t)dt$,
4. $D(F, G)$ *is integrable,*
5. $D(\int_I F(t)dt, \int_I G(t)dt) \leq \int_I D(F(t), G(t))dt$.

Definition 2.6 *[2] The mapping $F : I \to E^n$ is called a Hukuhara differential at $t_0 \in I$ if for some $h_0 > 0$ the Hukuhara differences*

$$F(t_0 + \Delta t) -_h F(t_0), F(t_0) -_h F(t_0 - \Delta t)$$

exist in E^n for all $0 < \Delta t < h_0$ and there exists an $F'(t_0) \in E^n$ such that

$$\lim_{\Delta t \to 0^+} D((F(t_0 + \Delta t) -_h F(t_0))/\Delta t, F'(t_0)) = 0$$

and

$$\lim_{\Delta t \to 0^+} D((F(t_0) -_h F(t_0 - \Delta t))/\Delta t, F'(t_0)) = 0$$

$F'(t)$ is called the Hukuhara derivative of F at t_0.

Definition 2.7 *[2]*

The mapping $F : I \to E^n$ is said to be differentiable at $t_0 \in I$ if, for any $\alpha \in [0, 1]$, the set-valued mapping $F_\alpha(t) = [F(t)]_\alpha$ is Hukuhara differentiable at point t_0 with $DF_\alpha(t_0)$ and the family $\{DF_\alpha(t_0) : \alpha \in [0, 1]\}$ denoted by fuzzy number $F(t_0) \in E^n$.

If the mapping $F : I \to E^n$ is differentiable at $t_0 \in I$, then we call $F'(t_0)$ the fuzzy derivative of $F(t)$ at the point t_0.

Theorem 2.8 *[2] Assume that $F : I \to E^n$ is differentiable. Let $F_\alpha(t) = [f_\alpha(t), g_\alpha(t)]$. Then f_α and g_α are differentiable and $[F'(t)]_\alpha = [f'_\alpha(t), g'_\alpha(t)]$ where f_α, g_α is α-cut level of the functions of f, g .*

Theorem 2.9 *[2] Suppose $F : I \to E^n$ is differentiable and suppose the derivative F' is integrable on I. Then, for each $s \in I$, we have*

$$F(s) = F(a) + \int_a^s F'(t)dt.$$

Definition 2.10 *[2] The mapping $f : I \times E^n \to E^n$ is said to be levelwise continuous at a point $(t_0, x_0) \in I \times E^n$ provided for any fixed $\alpha \in [0, 1]$ and arbitrary $\epsilon > 0$, there exists a $\delta(\epsilon, \alpha) > 0$ such that*

$$d([f(t, x)]_\alpha, [f(t_0, x_0)]_\alpha) < \epsilon$$

when $|t - t_0| < \delta(\epsilon, \alpha)$ and $d([x]_\alpha, [x_0]_\alpha) < \delta(\epsilon, \alpha)$ for all $t \in I$, $x \in E^n$.

Corollary 2.11 *[9]* Given that $F : I \times E^n \to E^n$ is continuous. Then the function

$$G(t) = \int_a^t F(s)ds, \ t \in I$$

is differentiable and $G'(t) = F(t)$. Now, if F is continuously differentiable on I, then we have the following mean value theorem

$$D(F(b), F(a)) \leq (b - a)sup\{D(F'(t), \tilde{0}), t \in I\}.$$

Hence, also have

$$D(G(b), G(a)) \leq (b - a)sup\{D(F(t), \tilde{0}), t \in I\}.$$

Theorem 2.12 *[2] Assume X to be a compact metric space and Y any metric space. The subset Ω of the space $C(X, Y)$ of continuous mapping of X into Y is totally bounded in the metric of uniform convergence if and only if Ω is equicontinuous on X, and $\Omega(x) = \{\phi(x) : \phi \in \Omega\}$ is a totally bounded subset of Y for each $x \in X$.*

Consider a delay differential system,

$$\frac{d}{dt}\big[x(t) + \sum_{i=1}^{l} P_i x(t - \tau_i)\big] + Q_0 x(t) + \sum_{j=1}^{n} Q_j x(t - \sigma_j) = 0 \qquad (1)$$

Definition 2.13 *[6] A solution of the system (1) $\boldsymbol{x}(t) = [x_1(t), ..., x_n(t)]$ is said to oscillate if every component $x_i(t)$ of the solution has arbitrarily large zeros. Otherwise it is called a non-oscillatory solution.*

Theorem 2.14 *[6] Assume that the coefficients P_i and Q_j of Eq. (1) are real $n \times n$ matrices and delays τ_i and σ_j are positive numbers. Let $\boldsymbol{x}(t)$ be a solution of Eq.(1) on $[0, \infty)$. Then there exist positive constant K and γ such that $\|\boldsymbol{x}(t)\| \leq Ke^{\gamma t}$ for $t \geq 0$.*

Theorem 2.15 *[6] Let $x \in C[[0, \infty), \Re]$ and suppose that there exist positive constants K and γ such that $|\boldsymbol{x}(t)| \leq Ke^{\gamma t}$ for $t \geq 0$. Then the abscissa of convergence σ_0 of the Laplace transform $X(s)$ of $\boldsymbol{x}(t)$ satisfies $\sigma_0 \leq \gamma$. Furthermore, X(s) exists and is an analytic function of s for $Res < \sigma_0$.*

Lemma 2.16 *[6] Consider the nonlinear delay differential system*

$$\dot{\boldsymbol{x}}(t) + \boldsymbol{f}(\boldsymbol{x}(t-\tau)) = 0. \tag{2}$$

Every non-oscillatory solution of Eq.(2) tends to zero as $t \to 0$.

Definition 2.17 *[6] A solution for the system (2) $\boldsymbol{x}(t) = [x_1(t), ..., x_n(t)]^T$ is said to oscillate if every component $x_i(t)$ of the solution has arbitrarily large zeros. Otherwise it is called a non-oscillatory solution.*

Theorem 2.18 *[6] Consider the delay differential system*

$$\dot{\boldsymbol{x}}(t) + p\boldsymbol{x}(t-\tau) = 0, \quad t \geq 0. \tag{3}$$

The following statements are equivalent:
(a) The delay differential system (3) has a positive solution.
(b) The delay differential inequality

$$\dot{\boldsymbol{y}}(t) + p\boldsymbol{y}(t-\tau) \leq 0, \quad t \geq 0 \tag{4}$$

has a positive solution.

Now, the existence and uniqueness theorem of delay differential equation will be given in crisp case.

Theorem 2.19 *[11] (Existence) Consider*

$$\begin{aligned} x'(t) &= f(t, x(t), x(t-\tau)) \\ x(\sigma) &= \Phi, \quad t \geq 0 \end{aligned} \tag{5}$$

Suppose ω is an open subset in $R \times C$ and f is continuous on ω. If $(\sigma, \Phi) \in \omega$, then there is a solution of (5)passing through (σ, Φ).

We say $f(t, \Phi)$ is Lipschitz in Φ in a compact set K of $R \times C$ if there is a constant $k > 0$ such that, for $(t, \Phi_i) \in K$, for $i = 1, 2 \mid f(t, \Phi_1) - f(t, \Phi_2) \mid \leq k \mid \Phi_1 - \Phi_2 \mid$.

Theorem 2.20 *[11] (Uniqueness)*
Suppose ω is an open set in $R \times C$, $f : \omega \to R^n$ is continuous, and $\boldsymbol{f}(t, \Phi)$ is Lipschitz in Φ in each set in ω. If $(\sigma, \Phi) \in \omega$, there is a unique solution of (5) through (σ, Φ).

3 Fuzzy Delay Predator-Prey (FDPP) System

In this section, we define a new system called fuzzy delay predator-prey system.

Consider the delay predator-prey system :

$$\begin{aligned} \dot{x}(t) &= x(1-x) - cyx \\ \dot{y}(t) &= cbe^{-d_j\tau}y(t-\tau)x(t-\tau) - dy \\ x(0) &= x_0 \\ y(0) &= y_0 \quad -\tau \le t \le 0. \end{aligned} \tag{6}$$

where x is prey a population, y is a predator population, d is the death rate of predator, c is the constant predator response, τ is the constant time necessary to change prey biomass into predator biomass and x_0,y_0 are initial conditions.

Then we fuzzify the linear part and $x(t), y(t)$ of the system (6) by using fuzzy symmetric triangular number and parametric form representation of α-cut and $x(t)$, $y(t)$ are non negative fuzzy functions as follows:

$\widetilde{1} = (1-(1-\alpha)\sigma_1, 1+(1-\alpha)\sigma_1)$

$\widetilde{d} = (d-(1-\alpha)\sigma_2, d+(1-\alpha)\sigma_2)$ where $0 \le \alpha \le 1$.

We can write the fuzzy delay predator-prey system in vector form:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{f}(t, \mathbf{x}(t), \mathbf{x}_t) \quad t \in J = [o, a], \\ \mathbf{x}(0) &= \mathbf{x}_0, \quad -\tau \le t \le 0 \end{aligned} \tag{7}$$

where

$$\mathbf{f}(t, \mathbf{x}(t), \mathbf{x}_t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}(t, \mathbf{x}(t), \mathbf{x}_t)$$

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & -d \end{bmatrix}, \quad \mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix},$$

$$\mathbf{B}(t, \mathbf{x}(t), \mathbf{x}_t) = \begin{bmatrix} -x^2 - cyx \\ cbe^{-d_j\tau}y(t-\tau)x(t-\tau) \end{bmatrix},$$

$$\mathbf{x}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}. \quad -\tau \le t \le 0$$

Where $\mathbf{f}$ is a fuzzy mapping from $E^2 \to E^2$, $\mathbf{x}(t)$ and $\mathbf{x}_t = \mathbf{x}(t-\tau)$ are non negative fuzzy functions of t, in E^2. The elements of matrix A are considered to be fuzzy numbers. $\dot{\mathbf{x}}(t)$ is the fuzzy derivative of $\mathbf{x}(t)$ and $\mathbf{x}_0$ is a fuzzy number.

4 Existence of the Solution

Definition 4.1 *The mapping $\boldsymbol{x}(t) : J \to E^2$ is called a solution to the problem (7) if it is levelwise continuous and satisfies the integral equation*

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 + \int_0^t \boldsymbol{f}(s, \boldsymbol{x}(s), \boldsymbol{x}_s)ds = \boldsymbol{x}_0 \tag{8}$$

$$+ \int_0^t (\boldsymbol{A}\boldsymbol{x}(s) + \boldsymbol{B}(s, \boldsymbol{x}(s), \boldsymbol{x}_s))ds \quad \forall t \in J.$$

Now let $Z = \{\xi \in E^2 : H(\xi, \mathbf{x}_0) \leq b\}$ be the space of continuous function with $H(\xi, \psi) = \sup_{0 \leq t \leq \gamma} D(\xi(t), \psi(t))$ and b a positive number.

We propose the existence and uniqueness theorem of the fuzzy delay predator-prey system (7) as follows.

Theorem 4.2 *Suppose that $\boldsymbol{A}$ and $\boldsymbol{B}$ are levelwise continuous on J implies that the mapping $\boldsymbol{f}\colon J \times Z \to E^2$ is levelwise continuous on J and there exists a constant G_0 such that*

$$D\Big(\boldsymbol{f}(t, \boldsymbol{x}(t), \boldsymbol{x}_t), \boldsymbol{f}(t, \boldsymbol{w}(t), \boldsymbol{w}_t)\Big) \leq D(\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{w})) \leq G_0 D(\boldsymbol{x}, \boldsymbol{w})$$

for all $\boldsymbol{x}, \boldsymbol{w} \in E^2$and $t \in J$.

Then there exists a unique solution $\boldsymbol{x}(t)$ of (7) defined on the interval $[0, \gamma]$ where

$$\gamma = \{a, b/M, 1/G_0\}$$

and

$$M = maxD\Big(\boldsymbol{f}(t, \boldsymbol{x}(t), \boldsymbol{x}_t), \tilde{0}\Big), \quad \tilde{0} \in E^2$$

Proof Consider the definition of the operator $\Phi : Z \to Z$ as

$$\Phi\mathbf{x}(t) = \mathbf{x}_0 + \int_0^t \mathbf{f}(t, \mathbf{x}(s), \mathbf{x}_s)ds = \tag{9}$$

$$\mathbf{x}_0 + \int_0^t (\mathbf{A}\mathbf{x}(s) + \mathbf{B}(t, \mathbf{x}(s), \mathbf{x}_s))ds$$

First, we prove that $\Phi : Z \to Z$ is continuous when $\xi \in Z$ and $H(\Phi\xi, \mathbf{x}_0) \leq b$. Since $\mathbf{f}$ is levelwise continuous, we put

$$M = maxD\Big(\mathbf{f}(t, \mathbf{x}(t), \mathbf{x}_t), \tilde{0}\Big)$$

$$
\begin{aligned}
&D(\Phi\xi(t+h),\Phi\xi(t)) = \qquad\qquad (10)\\
&D\Big(\mathbf{x}_0+\int_0^{t+h}\mathbf{f}(s,\xi(s),\xi_s)ds,\mathbf{x}_0+\int_0^{t}\mathbf{f}(s,\xi(s),\xi_s)ds\Big)\\
&\le D\Big(\int_0^{t+h}\mathbf{f}(s,\xi(s),\xi_s)ds,\int_0^{t}\mathbf{f}(s,\xi(s),\xi_s)ds\Big)\\
&\le \int_t^{t+h} D(\mathbf{f}(s,\xi(s),\xi_s),\tilde{0})ds\\
&= hM \to 0 \;\; as \;\; h\to 0.
\end{aligned}
$$

Hence, the mapping Φ is continuous. Now if

$$
D(\Phi\xi(t),\mathbf{x}_0) = D\Big(\mathbf{x}_0+\int_0^{t}\mathbf{f}(s,\xi(s),\xi_s)ds,\mathbf{x}_0\Big) \le \int_0^{t} D(\mathbf{f}(s,\xi(s),\xi_s),\tilde{0})ds = Mt
$$

and so

$$
H(\Phi\xi,\mathbf{x}_0) = \sup_{0\le t\le\gamma} D(\Phi\xi(t),\mathbf{x}_0) \le M\gamma \le b.
$$

Then Φ is a mapping from Z into Z. Since $C([0,\gamma],E^2)$ is a complete metric space with the metric H, we only need to prove that Z is a closed subset of $C([0,\gamma],E^2)$ so Z is a complete metric space. Suppose that ψ_n is a sequence in Z such that $\psi_n\to\psi\in C([0,\gamma],E^2)$ as $n\to\infty$. Then

$$
D(\psi(t),\mathbf{x}_0) \le D(\psi(t),\psi_n(t)) + D(\psi_n(t),\mathbf{x}_0),
$$

and also,

$$
H(\psi,\mathbf{x}_0) = \sup_{0\le t\le\gamma} D(\psi(t),\mathbf{x}_0) \le H(\psi,\psi_n) + H(\psi_n,\mathbf{x}_0) \le \epsilon + b
$$

for sufficiently large n and arbitrary $\epsilon>0$. Hence, $\psi\in Z$. This gives that Z is closed subset of $C([0,\gamma],E^2)$. Therefore Z is a complete metric space.

From Proposition (2.5) and the assumption of the theorem, we will prove that Φ is a contraction mapping. For $\xi,\psi\in Z$,

$$
\begin{aligned}
D(\Phi\xi(t),\Phi\psi(t)) &= D\Big(\mathbf{x}_0+\int_0^{t}\mathbf{f}(s,\xi(s),\xi_s)ds,\\
&\qquad \mathbf{x}_0+\int_0^{t}\mathbf{f}(s,\psi(s),\psi_s)ds\Big)\\
&\le \int_0^{t} D\Big(\mathbf{f}(s,\xi(s),\xi_s),\mathbf{f}(s,\psi(s),\psi_s)\Big)ds\\
&\le \int_0^{t} G_0 D(\xi(s),\psi(s))ds
\end{aligned}
$$

We conclude

$$H(\Phi\xi,\Phi\psi) \leq \sup_{t\in\gamma}\{\int_0^t G_0 D(\xi(s),\psi(s))ds\} \leq \gamma G_0 D(\xi(t),\psi(t)) \leq \gamma G_0 H(\xi,\psi).$$

Since $\gamma G_0 < 1$, Φ is a contraction mapping. Then Φ has a unique fixed point $\mathbf{x} \in C([0,\gamma],E^2)$ such that $\Phi\mathbf{x} = \mathbf{x}$, and

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_0^t \mathbf{f}(s,\mathbf{x}(s),\mathbf{x}_s)ds = \mathbf{x}_0 + \int_0^t (\mathbf{A}\mathbf{x}(s) + \mathbf{B}(s,\mathbf{x}(s),\mathbf{x}_s))ds.$$

Theorem 4.3 *Consider that* $\boldsymbol{f}$ *and* $\boldsymbol{x}_0$ *as in Theorem (4.2). And let* $\boldsymbol{x}(t,\boldsymbol{x}_0)$, $\boldsymbol{w}(t,\boldsymbol{w}_0)$ *be the solutions of the system (7) corresponding to* $\boldsymbol{x}_0$, $\boldsymbol{w}_0$, *respectively. Then there exists a constant* $q > 0$ *such that*

$$H(\boldsymbol{x}(t,\boldsymbol{x}_0),\boldsymbol{w}(t,\boldsymbol{w}_0)) \leq qD(\boldsymbol{x}_0,\boldsymbol{w}_0)$$

for any $\boldsymbol{x}_0,\boldsymbol{w}_0 \in E^2$ *and* $q = 1/(1-\gamma G_0)$.

Proof Assume that $\mathbf{x}(t,\mathbf{x}_0), \mathbf{w}(t,\mathbf{w}_0)$ are the solutions of the system (7) corresponding to $\mathbf{x}_0$, $\mathbf{w}_0$, respectively. Then

$$D\Big(\mathbf{x}(t,\mathbf{x}_0),\mathbf{w}(t,\mathbf{w}_0)\Big) = D\Big(\mathbf{x}_0 + \int_0^t \mathbf{f}(s,\mathbf{x}(s),\mathbf{x}_s)ds,$$

$$\mathbf{w}_0 + \int_0^t \mathbf{f}(s,\mathbf{w}(s),\mathbf{w}_s)ds\Big) \leq D(\mathbf{x}_0,\mathbf{w}_0)$$

$$+\int_0^t D\Big(\mathbf{f}(s,\mathbf{x}(s),\mathbf{x}_s),\mathbf{f}(s,\mathbf{w}(s),\mathbf{w}_s)\Big)ds \leq D(\mathbf{x}_0,\mathbf{w}_0) + \int_0^t G_0 D(\mathbf{x}(s),\mathbf{w}(s))ds$$

Therefore,

$$H\Big(\mathbf{x}(t,\mathbf{x}_0,\mathbf{w}(t,\mathbf{w}_0)\Big) \leq D(\mathbf{x}_0,\mathbf{w}_0) + (\gamma G_0)H(\mathbf{x}(t,\mathbf{x}_0),\mathbf{w}(t,\mathbf{w}_0)),$$

and

$$H(\mathbf{x}(t,\mathbf{x}_0),\mathbf{w}(t,\mathbf{w}_0)) \leq 1/(1-\gamma G_0)D(\mathbf{x}_0,\mathbf{w}_0).$$

Hence, the proof of the theorem is completed. Now, we give the generalization of Theorem 4.3 for fuzzy delay predator-prey system with initial value (7).

Theorem 4.4 *If* $\boldsymbol{f} : J \times E^2 \to E^2$ *is level-wise continuous and bounded, then the initial value problem (7) has at least one solution on the interval* J.

Proof The continuity and boundedness of $\mathbf{f}$ means that there exists an $r \leq 0$ such that

$$D(\mathbf{f}(t, \mathbf{x}(t), \mathbf{x}_t), \tilde{0}) \leq r, \quad t \in J, \ \mathbf{x} \in E^2.$$

Suppose that B is a bounded set in $C(J, E^2)$. The set $\Phi B = \{\Phi\mathbf{x} : \mathbf{x} \in B\}$ is totally bounded if and only if it is equicontinuous and for every $t \in J$, the set $\Phi B = \{\Phi\mathbf{x}(t) : t \in J\}$ is a totally bounded subset of E^2. For $t_0, t_1 \in J$ with $t_0 \leq t_1$, and $\mathbf{x} \in B$ we obtain,

$$D(\Phi\mathbf{x}(t_0), \Phi\mathbf{x}(t_1)) = D\Big(\mathbf{x}_0 + \int_0^{t_0} \mathbf{f}(s, \mathbf{x}(s), \mathbf{x}_s)ds, \mathbf{x}_0$$

$$+ \int_0^{t_1} \mathbf{f}(s, \mathbf{x}(s), \mathbf{x}_s)ds\Big) \leq D\Big(\int_0^{t_0} \mathbf{f}(s, \mathbf{x}(s), \mathbf{x}_s)ds, \int_0^{t_1} \mathbf{f}(s, \mathbf{x}(s), \mathbf{x}_s)ds\Big)$$

$$\leq \int_{t_0}^{t_1} D(\mathbf{f}(s, \mathbf{x}(s), \mathbf{x}_s), \tilde{0})ds \leq |t_1 - t_0| * sup\{D(\mathbf{f}(t, \mathbf{x}(t), \mathbf{x}_t), \tilde{0}) \ \ t \in J\} \leq |t_1 - t_0| * r.$$

This proves that ΦB is equicontinuous. Now, for fixed $t \in J$. We have

$$D(\Phi\mathbf{x}(t), \Phi\mathbf{x}(t^{'})) \leq |t - t^{'}| * r, \quad \text{for every } t^{'} \in J, \mathbf{x} \in B.$$

We conclude that the set $\{\Phi\mathbf{x} : \mathbf{x} \in B\}$ is totally bounded in E^2, and so ΦB is a relatively compact subset of $C(J, E^2)$. Therefore, Φ is compact, that is, Φ transforms bounded sets into relatively compact sets. We know that $\mathbf{x}$ is a fixed point of the operator Φ defined by equation (9) if and only if $\mathbf{x} \in C(J, E^2)$ is a solution of (7).

Now, for the metric space $(C(J, E^2), H)$, we consider the ball

$$B = \{\xi \in C(J, E^2), H(\xi, \tilde{0}) \leq m\}, \ \ m = a * r.$$

Thus, $\Phi B \subset B$. For $\mathbf{x} \in C(J, E^2)$,

$$D(\Phi\mathbf{x}(t), \Phi\mathbf{x}(0)) = D\Big(\mathbf{x}_0 + \int_0^{t} \mathbf{f}(s, \mathbf{x}(s), \mathbf{x}_s)ds, \mathbf{x}_0\Big)$$

$$\leq \int_0^{t} D(\mathbf{f}(s, \mathbf{x}(s), \mathbf{x}_s), \tilde{0})ds \leq |t| * r \leq a * r.$$

Therefore, we define $\tilde{0} : J \to E^2, \tilde{0}(t) = \tilde{0}, t \in J$ so, we have

$$H(\Phi\mathbf{x}, \Phi\mathbf{0}) = sup\{D(\Phi\mathbf{x}(t), \Phi\mathbf{0}(t)) : t \in J\}.$$

Hence, Φ is compact and consequently it has a fixed point $\mathbf{x} \in B$. This fixed point is a solution of the initial value problem (7).

5 Oscillation of Fuzzy Delay Predator-Prey System

This section deals with the oscillation of all solutions of fuzzy delay predator-prey system. Consider the system (7) we also list the following hypotheses on $\mathbf{f}$ which will be assumed only wherever this explicitly indicated:

$$\liminf_{\mathbf{u}\to 0} \frac{\mathbf{f}(\mathbf{u})}{\mathbf{u}} \geq 1, \tag{11}$$

$$\lim_{\mathbf{u}\to 0} \frac{\mathbf{f}(\mathbf{u})}{\mathbf{u}} = 1. \tag{12}$$

Whenever condition (11) or (12) is satisfied, the linear equation:

$$\dot{\mathbf{x}}(t) - C\mathbf{x}(t) - D\mathbf{x}(t-\tau) = 0 \tag{13}$$

will be called the linearized equation associated to the system (7). Where C, D are fuzzy matrices with characteristic equation

$$det(\lambda I - C - De^{-\lambda\tau}) = 0. \tag{14}$$

Now, in order to prove our oscillation theorem, we first assume that the theorem of the non-linear fuzzy delay differential system has the same oscillatory behavior as an associated linear system.

Theorem 5.1 *Suppose that every solution of linearized equation (13) is oscillatory. Then every solution of (7) also oscillates.*

Proof Assume for the sake of contradiction that Eq.(7) has a non-oscillatory solution $\mathbf{x}(t)$. We assume that $\mathbf{x}(t)$ is eventually positive. The case where $\mathbf{x}(t)$ is eventually negative is similar and will be omitted. By lemma (2.16) we know that $\lim_{t\to\infty} \mathbf{x}(t) = 0$. Thus by (11),

$\liminf_{t\to\infty} \frac{\mathbf{f}(\mathbf{x}(t-\tau))}{\mathbf{x}(t-\tau)} \geq 1.$

Let $\epsilon \in (0,1)$. Then there exists a T_ϵ such that for $t \geq T_\epsilon$ and $\mathbf{f}(\mathbf{x}(t-\tau)) \geq (1-\epsilon)\mathbf{x}(t-\tau)$. Hence from Eq.(7), $\dot{\mathbf{x}}(t) + (1-\epsilon)\mathbf{x}(t-\tau) \leq 0, \ \ t \geq T_\epsilon$.

It follows by Theorem (2.18) that Eq.(13) has positive solution. This contradicts the hypothesis that every solution of Eq.(13) is oscillatory and the proof is complete.

We give the oscillation theorem of the solution of linearized system.

Theorem 5.2 *Consider the linearized system (13) then the following statements are equivalent.*

(a) *Every solution of Eq. (13) oscillates componentwise.*

(b) *The characteristic equation (14) has no real roots.*

Proof The proof for $(a) \to (b)$ is simple. If λ_0 is a real root of characteristic equation (14) then there exists a non-zero vector $\mathbf{v}$ such that $(\lambda_0 I - C - De^{-\lambda_0 \tau})\mathbf{v} = 0$. Then clearly $\mathbf{x}(t) = e^{\lambda_0 t}\mathbf{v}$ is a solution of Eq.(13) with at least one non-oscillatory component.

For $(b) \to (a)$. Assume for the sake of contradiction that (b) holds and that the Eq. (13) has non-oscillatory solution $\mathbf{x}(t) = [x(t), y(t)]^T$. This means that one of the components of $\mathbf{x}(t)$ is non-oscillatory. And we consider the component $x(t)$ is positive for $t \geq \tau$. By Theorem 2.14 we know that $\mathbf{x}(t)$ is of exponential order and so there exists $\mu \in \Re$ such that the Laplace transform of $\mathbf{x}(t)$, $\mathbf{X}(s) = \int_0^\infty e^{-st}\mathbf{x}(t)dt$, exists for Re $s > \mu$. By taking Laplace transforms of both sides of Eq.(13) we obtain

$$\mathbf{F}(s)\mathbf{X}(s) = \psi(s), \ \ \text{Re } s > \mu \tag{15}$$

where

$$\mathbf{F}(s) = sI - C - De^{-s\tau} \tag{16}$$

and

$$\psi(s) = \mathbf{x}(0) - C - De^{-s\tau} \int_{-\tau}^{0} e^{-st}\mathbf{x}(t)dt. \tag{17}$$

By hypothesis, $det[\mathbf{F}(s)] \neq 0$ for all $s \in \Re$. Moreover,

$$\lim_{s\to\infty} (det[\mathbf{F}(s)]) = \infty \tag{18}$$

and so

$$det[\mathbf{F}(s)] > 0 \text{ for all } s \in \Re. \tag{19}$$

Let $X(s)$ be the Laplace transform of the first component $x(t)$ of the solution $\mathbf{x}(t)$. Then by Cramer's rule,

$$X(s) = \frac{det[\mathbf{M}(s)]}{det[\mathbf{F}(s)]}, \ \ Res > \mu \tag{20}$$

where

$$\mathbf{M}(s) = \begin{bmatrix} \psi_1(s) & F_{12}(s) \\ \psi_2(s) & F_{22}(s) \end{bmatrix}$$

ψ_i is the ith component of the vector $\psi(s)$ and $F_{ij}(s)$ is the (i, j)th component of the matrix $\mathbf{F}(s)$. Clearly, for all $i, j = 1, 2$ the functions $\psi_i(s)$ and $F_{ij}(s)$ are entire and hence $det[\mathbf{M}(s)]$ and $det[\mathbf{F}(s)]$ are also entire functions. Let σ_0

be the abscissa of convergence of $X(s)$ that is $\psi_0 = inf\{\sigma \in \Re : X(\sigma) \text{ exists}\}$. By using Theorem 2.15 we find $\sigma_0 = -\infty$ and (20) becomes

$$X(s) = \frac{det[\mathbf{M}(s)]}{det[\mathbf{F}(s)]} \text{ for all } s \in \Re. \tag{21}$$

As $x(t) > 0$, it follows that $X(s) > 0$ for all $s \in \Re$ and by (19) and (21), $det[\mathbf{M}(s)] > 0$ for $s \in \Re$. Now from the definition of $\mathbf{M}(s)$ and from (16) and (17) there are positive constants K, γ and s_0 such that

$$det[\mathbf{M}(s)] \leq Ke^{-\gamma s} \text{ for } s \leq -s_0. \tag{22}$$

Also from (18), (19) and the fact that $det[\mathbf{F}(s)]$ is a variable s and $e^{-s\tau}$, it follows that there exists a positive number m such that

$$det[\mathbf{F}(s)] \geq m \text{ for } s \in \Re. \tag{23}$$

From (21), (22) and (23) it follows that $X(s) = \int_0^\infty e^{-st}x(t)dt \geq \int_T^\infty e^{-st}x(s)dt \geq e^{-sT}\int_T^\infty x(t)dt > 0$

and so

$0 < \int_T^\infty x(t)dt \leq \frac{K}{m}e^{s(T-\gamma)} \to 0 \quad \text{as } s \to -\infty.$

This implies that $x(t) = 0$ for $t \geq T$, which is a contraction. The proof is complete.

6 Examples

Example 1

Consider the delay predator-prey system with initial value on $J = [0, 5]$:

$$\begin{aligned} \dot{x}(t) &= x - x^2 - yx \\ \dot{y}(t) &= -y + e^{-1}y(t-1)x(t-1) \\ x(0) &= x_0 \\ y(0) &= y_0 \quad -\tau \leq t \leq 0. \end{aligned} \tag{24}$$

The system (24) can be rewritten in vector form as follows:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(t, \mathbf{x}(t), \mathbf{x}_t) \\ \mathbf{x}(0) &= \mathbf{x}_0, \quad -\tau \leq t \leq 0 \end{aligned} \tag{25}$$

where $\mathbf{f}(t, \mathbf{x}(t), \mathbf{x}_t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}(t, \mathbf{x}(t), \mathbf{x}_t)$.

Therefore, $\mathbf{f}$ is a fuzzy mapping $\mathbf{f} : J \times E^2 \to E^2$. Since A is a fuzzy matrix, $\mathbf{x}(t)$, $\mathbf{x}_t = \mathbf{x}(t-1)$ are non negative fuzzy function of t in E^2 and $\mathbf{x}_0$ is a fuzzy number. Since A and B are levelwise continuous and bounded on J, the mapping $\mathbf{f}$ is levelwise continuous and bounded in E^2. By Theorem 4.4, $\mathbf{f}$ satisfies the conditions of Theorem 4.4 and hence the initial value problem (24) has a solution on J.

Consider the linearized system of (25) as follows:

$$\dot{\mathbf{x}}(t) - C\mathbf{x}(t) - D\mathbf{x}(t-\tau) = 0. \tag{26}$$

Where C, D are fuzzy matrices with the characteristic equation:

$$\begin{gathered}\lambda^4 + A\lambda^3 + B\lambda^2 + C\lambda + D + \\ e^{-(1+\lambda)}(E\lambda^3 + F\lambda^2 + G\lambda) + \\ e^{-2(1+\lambda)}(H\lambda^2 + I\lambda + J) = 0,\end{gathered} \tag{27}$$

where

$$\begin{gathered}A = -2a_1 + 4, \;\; B = a_1^2 - 4a_1 + 4(1-(1-\alpha)^2\sigma_1^2), \\ C = a_2^2, \;\; D = 2a_2^2(1+(1-\alpha)\sigma_1) - a_2^2 a_1, \\ E = -2, \;\; F = 4a_1 - 8, \\ G = -2a_1^2 + 8a_1 - 8(1-(1-\alpha)^2\sigma_1^2), \\ H = (1-(1-\alpha)^2\sigma_1^2), \\ I = -2a_1(1-(1-\alpha)^2\sigma_1^2) + 4(1-(1-\alpha)^2\sigma_1^2), \\ J = a_1^2(1-(1-\alpha)^2\sigma_1^2) - 4a_1(1-(1-\alpha)^2\sigma_1^2) + \\ 4(1-(1-\alpha)^2\sigma_1^2)^2.\end{gathered} \tag{28}$$

The characteristic equation (27) has no real roots. Therefore, by Theorem 5.2 the linearized system oscillates. And by Theorem 5.1 the system (25) also oscillates.

Example 2

Let the delay predator-prey system have initial value on $J = [0, 4]$:

$$\begin{gathered}\dot{x}(t) = x - x^2 - 2yx \\ \dot{y}(t) = -y + 2e^{-4}y(t-2)x(t-2) \\ x(0) = x_0 \\ y(0) = y_0 \quad -\tau \le t \le 0.\end{gathered} \tag{29}$$

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(t, \mathbf{x}(t), \mathbf{x}_t)\\ \mathbf{x}(0) &= \mathbf{x}_0, \quad -\tau \le t \le 0\end{aligned} \tag{30}$$

where $\mathbf{f}(t, \mathbf{x}(t), \mathbf{x}_t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}(t, \mathbf{x}(t), \mathbf{x}_t)$.

Then $\mathbf{f} : I \times E^2 \to E^2$ is a fuzzy mapping. Since $\mathbf{A}$ is a fuzzy matrix on E^2 and $\mathbf{x}(t)$, $\mathbf{x}(t-2)$ are non negative fuzzy function on t in E^2 and $\mathbf{x}_0$ is a fuzzy number. Since E^2 is a bounded space so, $\mathbf{A}, \mathbf{B}$ are bounded on E^2 and are also levelwise continuous. Hence $\mathbf{f}$ satisfies the assumptions of the Theorem 4.4 and the initial value problem (30) has a solution on J.

Consider the linearized system of (30) as follows:

$$\dot{\mathbf{x}}(t) - C\mathbf{x}(t) - D\mathbf{x}(t-\tau) = 0. \tag{31}$$

Where C, D are fuzzy matrices with characteristic equation for (31):

$$\lambda^4 + A\lambda^3 + B\lambda^2 + C\lambda + D + \tag{32}$$

$$e^{-2(2+\lambda)}(E\lambda^3 + F\lambda^2 + G\lambda) +$$

$$e^{-4(2+\lambda)}(H\lambda^2 + I\lambda + J) = 0,$$

where

$$A = -2a_1 + 4, \;\; B = a_1^2 - 4a_1 + 4(1 - (1-\alpha)^2\sigma_1^2), \tag{33}$$

$$C = a_2^2, \;\; D = 2a_2^2(1 + (1-\alpha)\sigma_1) - a_2^2 a_1,$$

$$E = -4, \;\; F = 8a_1 - 16,$$

$$G = -4a_1^2 + 16a_1 - 16(1 - (1-\alpha)^2\sigma_1^2),$$

$$H = 4(1 - (1-\alpha)^2\sigma_1^2),$$

$$I = -8a_1(1 - (1-\alpha)^2\sigma_1^2) + 16(1 - (1-\alpha)^2\sigma_1^2),$$

$$J = 4a_1^2(1 - (1-\alpha)^2\sigma_1^2) - 16a_1(1 - (1-\alpha)^2\sigma_1^2) +$$

$$16(1 - (1-\alpha)^2\sigma_1^2)^2.$$

The characteristic equation (**??**) has no real roots. Therefore, by Theorem 5.2 the linearized system oscillates and by Theorem 5.1 the system (30) also oscillates.

7 Conclusion

In this paper, we proposed a system of fuzzy delay predator-prey system. We successfully proved the existence and uniqueness solution of the FDPP on the interval $[0, \gamma]$. We also generalized Theorem 4.2 for Existence Theorem of the solution of FDPP system with fuzzy initial condition on J. Also, we introduced the oscillation theorem of solutions of fuzzy predator-prey system. The applicability of the results are demonstrated by the examples given.

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