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The Solution of Euler-Cauchy Equation Using Laplace Transform

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Abstract

Euler-Cauchy equation is a typical example of ODE with variable coefficients. Since this equation has a simple form, we would like to start from this equation to find the solution of ODEs with variable coefficients. In this article, we have checked the solution of Euler-Cauchy equation by using Laplace transform. The proposed formula can be applied to another ODEs with variable coefficients, and another integral transforms are can be done as well in a similar way.

Mathematics Subject Classification: 44A10, 34A30

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1 Introduction

It has been pursuing effort to find the solution of ODEs with variable coefficients by using integral transforms. However, Anyone could not find the glossy method so far. Hence, we have tried this topic, and we found a somewhat reasonable method for this problem.

To begin with, let us see the form of Euler-Cauchy equations. The equations are ODEs of the form $t^2y'' + aty' + by = 0$ with given constants a and b and unknown $y(t)$, and this equation is a fundamental ODE with variable coefficients[14]. Additionally, it is well-known fact that this equation has the auxiliary equation $m^2+(a-1)m+b=0$ for a trial solution $y=t^m$. To extend this form to order n, the form would become

$$a_nt^ny^{(n)}(t) + a_{n-1}t^{n-1}y^{(n-1)}(t) + \cdots + a_0y(t) = 0$$

for $y^{(n)}(t)$ is the n-th derivative of the function $y(t)$. The most common form is the second-order equation, and it appears in solving Laplace equation in a polar coordinates, describing time-harmonic vibrations of a thin elastics rod, boundary value problem in spherical coordinates and so on[10].

With relation to this topic, several researches have been pursued. The researches with respect to differential equations with variable coefficients appear in [1, 3, 5, 6-10, 15-17, 19-20], and we can find the contents with respect to integral transforms in [2, 4, 11-13, 18].

Since this equation has an important meaning to study ODEs with variable coefficients as a base, we would like to start from the equation in the study with respect to it. Using the property of $\pounds(tf(t)) = -F'(s))$ for $\pounds(f) = F(s)$, we have found the formula to find the solution of Euler-Cauchy equation $t^2y'' + aty' + by = 0$ by using Laplace transform in theorem 2.1. Additionally, we have checked the case of the third-order as well.

2 The solution of Euler-Cauchy equation by using Laplace transform

We would like to check the solution of Euler-Cauchy equation by using Laplace transform. The fundamental representation with respect to the equation can be found in our previous article[10].

Theorem 2.1 *Let $Y = \pounds(y) = F(s)$. Then the solution of Euler-Cauchy equation $t^2y'' + aty' + by = 0$ can be represented by $y = \pounds^{-1}(s^m)$ where,*

$$m = \frac{a-3 \pm \sqrt{(a-1)^2-4b}}{2} \qquad (*)$$

for $Y = s^m$.

Proof. Taking Laplace transform on both sides, by [12], we have

$$s^2\frac{d^2Y}{ds^2} + (4-a)s\frac{dY}{ds} + (b-a+2)Y = 0$$

for $Y = \pounds(y) = F(s)$. Since Y is a function of s, let us put $Y = s^m$ as a trial solution for m is a constant. Then we have $dY/ds = ms^{m-1}$ and $d^2Y/ds^2 = m(m-1)s^{m-2}$, and the given equation becomes

$$m(m-1)s^m + (4-a)ms^m + (b-a+2)s^m = 0.$$

Dropping a common factor s^m, we have

$$m(m-1) + (4-a)m + (b-a+2) = 0.$$

Organizing this equality, we have

$$m^2 + (3-a)m + b - a + 2 = 0.$$

Hence,

$$m = \log_s Y = \frac{a-3 \pm \sqrt{(a-1)^2 - 4b}}{2}$$

and the solution

$$y = \pounds^{-1}(Y) = \pounds^{-1}(s^{\frac{a-3\pm\sqrt{(a-1)^2-4b}}{2}})$$

for $\pounds(y) = F(s)$.

Example 2.2 *The Euler-Cauchy equation $t^2y'' - 3ty' + 3y = 0$ has a solution $y = c_1t + c_2t^3$ for c_1 and c_2 are arbitrary constant.*

Solution. Since $a = -3$ and $b = 3$, we substitute this into the equation $(*)$. This gives $Y = s^{-2}$ and $Y = s^{-4}$. Thus, by the table of Laplace transform, we have the basis $y = t$ and $y = \frac{1}{6}t^3$. Implies, the general solution corresponding to this basis is $y = c_1t + c_2t^3$.

In the above example, we note that $\pounds(t^n) = n!/s^{n+1}$ for $Y = \pounds(f) = F(s)$.

Example 2.3 *The Euler-Cauchy equation $t^2y'' - ty' + y = 0$ has a general solution $y = (c_1 + c_2 \ln t)t$.*

Solution. Since $a = -1$ and $b = 1$, by $(*)$, $Y = s^{-2}$ and so, we have $y = t$. Thus, the general solution is $y = (c_1 + c_2 \ln t)t$.

If we define this formula at $n = -1$, we have

$$\pounds(1/t) = -1 \qquad (**)$$

for $(-1)! = -1$. This formula has a high applicability in the use of theorem 2.1.

Example 2.4 *The Euler-Cauchy equation* $t^2y'' + 1.5ty' - 0.5y = 0$ *has a general solution* $y = c_1\sqrt{t} + c_2/t$.

Solution. Since $a = 1.5$ and $b = -0.5$, by $(*)$ and $(**)$, $Y = 1$ and $Y = s^{-3/2}$. Since $\mathcal{L}(2\sqrt{t/\pi}) = 1/s^{3/2}$, we have $y = -1/t$ and $y = 2\sqrt{t/\pi}$. Thus, the general solution is

$$y = c_1\sqrt{t} + c_2/t.$$

Let us see the another example appearing in [14].

Example 2.5 *Find the electrostatic potential* $v = v(r)$ *between two concentric spheres of radii* $r_1 = 5$ *and* $r_2 = 10$ *kept at potentials* $v_1 = 110V$ *and* $v_2 = 0$*, respectively.*

Solution. Since the electrostatic potential $v(r)$ is a solution of the Euler-Cauchy equation $rv'' + 2v' = 0$, by multiplying r on both sides, we have $a = 2$ and $b = 0$. By $(*)$ and $(**)$, $Y = 1$ and $Y = 1/s$. Hence, we have $y = -1/r$ and $y = 1$. This gives the general solution

$$y = c_1 + c_2/r.$$

From the boundary conditions, we can easily obtain the solution $v(r) = -110 + 1100/r$.

Note that $Y = e^{ms}$ is not appropriate as a trial solution because m contains a variable s. To expatiate on this, if $Y = e^{ms}$, then m is denoted as

$$m = \frac{a - 4 \pm \sqrt{(a-2)^2 + 4(1-b)}}{2s}.$$

Hence, we cannot find the solution by this trial solution e^{ms}, and it is just used for change of variable in the equation. While, $Y = s^m$ has validity as a trial solution of Euler-Cauchy equation because Y is just denoted by constants.

Next, we would like to check the third-order Euler-Cauchy equation. We consider the equation

$$f(t) = t^3y''' + at^2y'' + bty' + cy = 0$$

where a and b are constants. Taking Laplace transform, we have $\mathcal{L}(f) = F(s) = Y$ for $Y = s^m$. Hence, by the inverse transform, we can obtain $y = f(t)$ and so, a general solution can be done as well.

Lemma 2.6 *Formula $\pounds(ty(t)) = -F'(s)$ can be used to solve certain differential equations with variable coefficients.*

$$\begin{aligned}
&(1)\quad \pounds(ty''') = -s^3\frac{dY}{ds} - 3s^2Y + 2y(0)s + y'(0).\\
&(2)\quad \pounds(t^2y''') = s^3\frac{d^2Y}{ds^2} + 6s^2\frac{dY}{ds} + 6Ys - 2y(0).\\
&(3)\quad \pounds(t^3y''') = -s^3\frac{d^3Y}{ds^3} - 9s^2\frac{d^2Y}{ds^2} - 18s\frac{dY}{ds} - 6Y.
\end{aligned}$$

Proof. If the Laplace transform of $y(t)$ exists on $t \geq 0$(i.e., y is continuous and satisfies the growth restriction), the transform of $\pounds(y''')$ is represented by

$$\pounds(y''') = s^3\pounds(y) - s^2y(0) - sy'(0) - y''(0).$$

Using this equation and by the simple calculation, we can obtain the above results.

Theorem 2.7 *The solution of Euler-Cauchy equation $t^3y''' + at^2y'' + bty' + cy = 0$ can be represented by $Y = s^m$, where m satisfies the equation*

$$m^3 + (6-a)m^2 + (b-3a+11)m + (b-2a-c+6) = 0 \qquad (***)$$

for $Y = \pounds(y) = F(s)$.

Proof. Let us denote $Y = \pounds(y) = F(s)$. Taking Laplace transform on both sides of the given equation and organizing the equation, we have

$$s^3\frac{d^3Y}{ds^3} + (9-a)s^2\frac{d^2Y}{ds^2} + (b-4a+18)s\frac{dY}{ds} + (b-2a-c+6)Y = 0$$

because of lemma 2.6. Let us put $Y = s^m$. By the similar way with theorem 2.1, we have

$$m^3 + (6-a)m^2 + (b-3a+11)m + (b-2a-c+6) = 0$$

for constant m. Hence, $y = \pounds^{-1}(Y)$ gives a general solution.

If so, let us check an example with respect to theorem 2.7.

Example 2.8 *The Euler-Cauchy equation $t^3y''' - 3t^2y'' + 6ty' - 6y = 0$ has a general solution $y = c_1t + c_2t^2 + c_3t^3$.*

Solution. By the equation $(***)$, we have

$$m^3 + 9m^2 + 26m + 24 = 0$$

for $Y = s^m$. By the factorization, we have

$$(m + 2)(m + 3)(m + 4),$$

and so, $m = -4, -3, -2$. Thus, $Y = s^{-4}$, s^{-3} and s^{-2}. Since $y = \pounds^{-1}(Y)$, we have the bases $y = \frac{1}{6}t^3$, $\frac{1}{2}t^3$ and t. Hence, we conclude that a general solution of the given equation is

$$y = c_1 t + c_2 t^2 + c_3 t^3.$$

Remark. Since the Laplace transform does not deal with the form of S^n for n is positive, these formulas does not as well.

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