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Maximal Zero-Subspaces of Diagonal Polynomials on Banach Spaces

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Abstract

We consider questions related to linear subspaces of diagonal polynomials on Banach spaces, their properties and maximal subspaces.

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1 Introduction and Preliminaries

Zero-sets of complex polynomials in $\mathbb{C}^n$ is a standart topic of Classical Algebraic Geometry. Zeros of polynomials on infinite dimensional Banach spaces was studied by many authors, for example R. Aron, C.Boyd, R. Ryan, I. Zalduendo, S. Todorcevic, M. Fernandez-Unzeueta, A. Zagorodnyuk, A. Plichko, R. Gonzalo, J. Ferrer, P. Hajek and others (see a survey [1]).

In [2] it was proved that zero-set of an arbitrary infinite dimensional polynomial contains an infinite dimensional linear subspace. Some questions related to Aron-Berner extensions of polynomials on infinite dimensional complex Banach spaces using natural extensions of their zeros was considered in [3].

In this paper we consider questions related to linear subspaces of diagonal polynomials on Banach spaces, prove some properties of such subspaces and investigate their maximality.

Let X be a Banach space. A map $P_n : X \to \mathbb{C}$ is an n-degree continuous homogeneous polynomials, $n \geqslant 1$ (n-homogeneous polynomial), if there is an n-linear mapping $B_n : \underbrace{X \times X \ldots X}_{n} \to \mathbb{C}$ such that $P_n(x) = B_n(x, x, \ldots, x)$ for all $x \in X$.

In the paper we consider the case when X is a separable Banach space with unconditional basis $\{e_n\}$.

Definition 1.1. *An n-homogeneous continuous polynomial of the form* $P(x) = \sum_{n=1}^{\infty} c_k x_k^n$, *where* c_k *is a bounded sequence is called* ***a diagonal polynomial****.*

Typical examples of diagonal polynomials are polynomials of the form $F_n(x) = \sum_{n=1}^{\infty} x_k^n$, where $x = \sum_{n=1}^{\infty} x_n e_n \in \ell_p, p \leqslant n$.

2 Main Results

Let $v_l(i) = e_{2l-1} + \lambda_i e_{2l}$, $l = 1, \ldots, n$, where $\lambda_i, i = 1, \ldots, n$ is one of n-th roots of –1. Next we construct spaces V_i, such that

$$V_i = \overline{\text{span}}\{v_1(i), \ldots, v_n(i), \ldots\}.$$

If $y_i \in V_i$, then $y_i = \sum_{k=1}^{\infty} a_k v_k(i)$.

Proposition 2.1. *Each of these spaces* V_i*,* $i = 1, \ldots, n$ *is contained in the zero-set of* F_n *and* $V_i \cap V_j = \{0\}$ *for all* $i, j = 1, \ldots, n$*,* $i \neq j$*.*

Proof. If $y_i \in V_i$ for $i = 1, \ldots, n$, then

$$F_n(y_i) = F_n\left(\sum_{k=1}^{\infty} a_k v_k(i)\right) = F_n\left(\sum_{k=1}^{\infty} a_k (e_{2k-1} + \lambda_i e_{2k})\right) =$$

$$= \sum_{k=1}^{\infty} (a_k^n + \lambda_i^n a_k^n) = \sum_{k=1}^{\infty} (a_k^n + (-1) a_k^n) = 0.$$

So, $V_i \subset \text{Ker} F_n$.

Let $y \in V_i$ and $y \in V_j$, $i \neq j$ and $y \neq 0$. Then

$$y = \sum_{k=1}^{\infty} a_k v_k(i) = \sum_{m=1}^{\infty} b_m v_m(j) = \sum_{n=1}^{\infty} y_n e_n,$$

for some $v_k(i) = e_{2k-1} + \lambda_i e_k$, $v_m(j) = e_{2m-1} + \lambda_j e_{2m}$, $i, j = 1, \ldots n$, λ_i and λ_j are different n-th roots of –1.

Since

$$y = \sum_{k=1}^{\infty} a_k(e_{2k-1} + \lambda_i e_{2k}) = \sum_{n=1}^{\infty} y_n e_n,$$

then for every k such that $a_k \neq 0$ we have the next equations:

$$\begin{cases} y_{2k-1} = a_k, \\ y_{2k} = \lambda_i a_k. \end{cases} \tag{1}$$

On the other hand, for every m such that $b_m \neq 0$ and

$$y = \sum_{m=1}^{\infty} b_m(e_{2m-1} + \lambda_j e_{2m}) = \sum_{n=1}^{\infty} y_n e_n,$$

we have another equations:

$$\begin{cases} y_{2m-1} = b_m, \\ y_{2m} = \lambda_j b_m. \end{cases} \tag{2}$$

If $m = k$, then from (1) and (2) it follows that

$$\begin{cases} a_k = b_m, \\ \lambda_i a_k = \lambda_j b_m \end{cases}$$

and hence $\lambda_i = \lambda_j$. A contradiction.

So, $V_i \cap V_j = \{0\}$ that was needed to show. □

Let us denote by $A_{F_n} : \underbrace{X \times X \times \ldots \times X}_{n} \to \mathbb{C}$ the n-linear symmetric form associated with polynomial F_n. $A_{F_n}(x, \ldots, x) = F_n(x)$ and

$$A_{F_n}(x_1, x_2 \ldots, x_n) = \sum_{k=1}^{\infty} x_{1k} x_{2k} \cdot \ldots \cdot x_{nk},$$

where $x_j = (x_{j1}, x_{j2}, \ldots x_{jn}, \ldots) = \sum\limits_{k=1}^{\infty} x_{jk} v_k(j)$.

Proposition 2.2. *For every diagonal polynomial* F_n

$$A_{F_n}(x_1, x_2, \ldots, x_n) = 0,$$

where $x_m \in V_m$, $m = 1, \ldots n$.

Proof. From direct calculations we have

$$A_{F_n}(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{\infty} x_{1_k} x_{2_k} \ldots x_{n_k} =$$

$$= A_{F_n}\left(\sum_{k=1}^{\infty} x_{1k}v_k(1), \sum_{k=1}^{\infty} x_{2k}v_k(2), \dots, \sum_{k=1}^{\infty} x_{nk}v_k(n)\right) =$$

$$= A_{F_n}\left(\sum_{k=1}^{\infty} x_{1k}(e_{2k-1}+\lambda_1 e_{2k}), \sum_{k=1}^{\infty} x_{2k}(e_{2k-1}+\lambda_2 e_{2k}), \dots, \sum_{k=1}^{\infty} x_{nk}(e_{2k-1}+\lambda_n e_{2k})\right) =$$

$$= A_{F_n}\left(\sum_{k=1}^{\infty} x_{1k}e_{2k-1}, \sum_{k=1}^{\infty} x_{2k}e_{2k-1}, \dots, \sum_{k=1}^{\infty} x_{nk}e_{2k-1}\right) +$$

$$+ A_{F_n}\left(\sum_{k=1}^{\infty} x_{1k}e_{2k-1}, \sum_{k=1}^{\infty} x_{2k}\lambda_2 e_{2k}, \sum_{k=1}^{\infty} x_{3k}e_{2k-1} \dots, \sum_{k=1}^{\infty} x_{nk}e_{2k-1}\right) +$$

$$+ \dots + A_{F_n}\left(\sum_{k=1}^{\infty} x_{1k}e_{2k-1}, \sum_{k=1}^{\infty} x_{2k}e_{2k-1}, \dots, \sum_{k=1}^{\infty} x_{nk}\lambda_n e_{2k}\right) +$$

$$+ A_{F_n}\left(\sum_{k=1}^{\infty} x_{1k}\lambda_1 e_{2k}, \sum_{k=1}^{\infty} x_{2k}e_{2k-1}, \dots, \sum_{k=1}^{\infty} x_{nk}e_{2k-1}\right) +$$

$$+ A_{F_n}\left(\sum_{k=1}^{\infty} x_{1k}\lambda_1 e_{2k}, \sum_{k=1}^{\infty} x_{2k}\lambda_2 e_{2k}, \sum_{k=1}^{\infty} x_{3k}e_{2k-1} \dots, \sum_{k=1}^{\infty} x_{nk}e_{2k-1}\right) +$$

$$+ \dots + A_{F_n}\left(\sum_{k=1}^{\infty} x_{1k}\lambda_1 e_{2k}, \sum_{k=1}^{\infty} x_{2k}\lambda_2 e_{2k}, \dots, \sum_{k=1}^{\infty} x_{nk}\lambda_n e_{2k}\right).$$

Since all summands in the right side, excepting the first one and the last one terms are equal to zero, we have

$$A_{F_n}(x_1, x_2, \dots, x_n) = A_{F_n}\left(\sum_{k=1}^{\infty} x_{1k}v_k(1), \sum_{k=1}^{\infty} x_{2k}v_k(2), \dots, \sum_{k=1}^{\infty} x_{nk}v_k(n)\right) =$$

$$= A_{F_n}\left(\sum_{k=1}^{\infty} x_{1k}e_{2k-1}, \sum_{k=1}^{\infty} x_{2k}e_{2k-1}, \dots, \sum_{k=1}^{\infty} x_{nk}e_{2k-1}\right) +$$

$$+ A_{F_n}\left(\sum_{k=1}^{\infty} x_{1k}\lambda_1 e_{2k}, \sum_{k=1}^{\infty} x_{2k}\lambda_2 e_{2k}, \dots, \sum_{k=1}^{\infty} x_{nk}\lambda_n e_{2k}\right) =$$

$$= \sum_{k=1}^{\infty} x_{1k}x_{2k}\dots x_{nk} + \sum_{k=1}^{\infty} \underbrace{\lambda_1\lambda_2\dots\lambda_n}_{=-1} x_{1k}x_{2k}\dots x_{nk} =$$

$$= \sum_{k=1}^{\infty} x_{1k}x_{2k}\dots x_{nk} + (-1)\sum_{k=1}^{\infty} x_{1k}x_{2k}\dots x_{nk} = 0.$$

Hence $A_{F_n}(x_1, x_2, \dots, x_n) = 0$. □

Proposition 2.3. *Each subspace V_l, $l = 1, \dots, n$ is complemented in ℓ_p, $p \leqslant n$ and $V_i \bigoplus V_j = \ell_p$, for all $i, j = 1 \dots, n$, $i \neq j$.*

Proof. In order to prove that $V_i \bigoplus V_j = \ell_p$, for all $i, j = 1 \dots, n$, $i \neq j$ we have to show that $V_i \bigoplus V_j$ contains all the basic vectors e_n.

So basic vectors can be obtained by

$$e_{2k} = \frac{v_k(i) - v_k(j)}{\lambda_i - \lambda_j},$$

$$e_{2k-1} = \frac{\lambda_j v_k(i) - \lambda_i v_k(j)}{\lambda_j - \lambda_i}$$

and clearly belong to $V_i \bigoplus V_j$.

We have shown that in $V_i \bigoplus V_j$ there are all basis vectors e_n of ℓ_p, thus $V_i \bigoplus V_j = \ell_p$. Hence the space V_i, $i = 1, \dots, n$ is completable to V_j, $j = 1, \dots, n$ in ℓ_p for all $i, j = 1, \dots, n$, $i \neq j$. □

Let us denote by w_m vectors constructed according to the formula:

$$w_m = e_{2m-1} + c_1^{\frac{1}{n}} e_{2m} + c_2^{\frac{1}{n}} e_{2m+1}, \ m = 1, \dots, n,$$

where $1 + c_1 + c_2 = 0$. Using these vectors we can construct spaces W_m, such that

$$W_m = \overline{\text{span}}\{w_1, \dots, w_n, \dots\}.$$

If $y_m \in W_m$, then $y_m = \sum\limits_{m=1}^{\infty} a_m w_m$.

Proposition 2.4. *Each of these spaces W_m, $m = 1, \dots, n$ is contained in the zero-set of the polynomial F_n, that is $W_m \subset \ker F_n$.*

Proof. If $y_m \in W_m$ for $m = 1, \dots, n$, then

$$F_n(y_m) = F_n\left(\sum_{m=1}^{\infty} a_m w_m\right) = F_n\left(\sum_{m=1}^{\infty} a_m (e_{2m-1} + c_1^{\frac{1}{n}} e_{2m} + c_2^{\frac{1}{n}} e_{2m+1})\right) =$$

$$= \sum_{m=1}^{\infty} (a_m^n + (c_1^{\frac{1}{n}} a_m)^n + (c_2^{\frac{1}{n}} a_m)^n) = \sum_{m=1}^{\infty} a_m^n (\underbrace{1 + c_1 + c_2}_{=0}) = 0.$$

Since $F_n(y_m) \subset \ker F_n$ for all $y_m \in W_m$, $m = 1, \dots, n$, then $W_m \subset \ker F_n$. □

Proposition 2.5. *Any vector $w_m \in W_m$, $m = 1, \dots, n$ can not be represented as a linear combination of vectors $v_m(i) \in V_i$, $m, i = 1, \dots, n$ and hence $W_m \not\subseteq V_i$.*

Proof. Let us suppose that $w_m = a_1 v_m(1) + a_2 v_m(2)$. Then

$$e_{2m-1} + c_1^{\frac{1}{n}} e_{2m} + c_2^{\frac{1}{n}} e_{2m+1} = a_1(e_{2m-1} + \lambda_1 e_{2m}) + a_2(e_{2m+1} + \lambda_2 e_{2m+2});$$

$$e_{2m-1} + c_1^{\frac{1}{n}} e_{2m} + c_2^{\frac{1}{n}} e_{2m+1} + 0 \cdot e_{2m+2} = a_1 e_{2m-1} + a_1\lambda_1 e_{2m} + a_2 e_{2m+1} + a_2\lambda_2 e_{2m+2}.$$

We have the next equations:

$$\begin{cases} a_1 = 1, \\ a_1\lambda_1 = c_1^{\frac{1}{n}}, \\ a_2 = c_2^{\frac{1}{n}}, \\ a_2\lambda_2 = 0, \end{cases} \Rightarrow \begin{cases} \lambda_1 = c_1^{\frac{1}{n}}, \\ \lambda_2 = 0. \end{cases}$$

This is a contradiction, because $\lambda_2 \neq 0$. Hence vectors w_m do not belong to a linear combination of vectors $v_m(i)$ and $W_m \nsubseteq V_i$. □

Proposition 2.6. *The space W_m is a maximal for the case $n = 2$.*

Proof. Let $y = \sum_{i=1}^{\infty} a_i e_i$ be an arbitrary vector, such that $F_2(y) = 0$, i.e. $\sum_{i=1}^{\infty} a_i^2 = 0$.

Let $y_0 = e_1 + c_1^{\frac{1}{2}} e_2 + c_2^{\frac{1}{2}} e_3 = (1, c_1^{\frac{1}{2}}, c_2^{\frac{1}{2}}, 0, \ldots, 0) \in W_m$. We will seek the vector y such that $F_2(y_0 + y) = 0$.

Let us choose $y = (a_1, a_2, a_3, 0, \ldots, 0)$ and $a_1, a_2, a_3 \neq 0$.

$$F_2(y_0 + y) = (1 + a_1)^2 + (c_1^{\frac{1}{2}} + a_2)^2 + (c_2^{\frac{1}{2}} + a_3)^2 = 1 + 2a_1 + a_2 + c_1 +$$
$$+ 2\sqrt{c_1} a_2 + a_2^2 + c_2 + 2\sqrt{c_2} a_3 + a_3^2 = 2(a_1 + \sqrt{c_1} a_2 + \sqrt{c_2} a_3).$$

So, we find equations:

$$\begin{cases} a_1^2 + a_2^2 + a_3^2 = 0, \\ a_1 + \sqrt{c_1} a_2 + \sqrt{c_2} a_3 = 0 \end{cases}$$

and we have that

$$a_2 = a_3 \sqrt{\frac{c_1}{c_2}},$$

$$a_1 = \sqrt{-a_2^2 - a_3^2} = \sqrt{-a_3^2 \frac{c_1}{c_2} - a_3^2} = a_3 \sqrt{-\frac{c_1}{c_2} - 1} = a_3 \sqrt{\frac{-c_1 - c_2}{c_2}} = \frac{a_3}{\sqrt{c_2}}.$$

Thus

$$y = (a_3 \frac{1}{\sqrt{c_2}}, -a_3 \sqrt{\frac{c_1}{c_2}}, a_3, 0, \ldots, 0) = a_3 \frac{1}{\sqrt{c_2}} (1, \sqrt{c_1}, \sqrt{c_2}, 0, \ldots, 0) \in W_m.$$

Since the vector y was chosen arbitrarily, it means that the space W_m is maximal. □

Let us denote by z_k vectors constructed according to the formula:

$$z_k = e_{2k-1} + c_1^{\frac{1}{n}} e_{2k} + c_2^{\frac{1}{n}} e_{2k+1} + c_3^{\frac{1}{n}} e_{2k+2},\ k = 1, \ldots, n,$$

where $1 + c_1 + c_2 + c_3 = 0$. Using these vectors we construct spaces Z_k, such that

$$Z_k = \overline{\operatorname{span}}\{z_1, \ldots, z_n, \ldots\}.$$

Similarly as for space W_m for space Z_k we can shown that:

1. $Z_m \subset \ker F_n$;
2. $Z_m \subsetneq V_i$ and $Z_m \subsetneq W_m$.

Proposition 2.7. *The space Z_k is not maximal.*

Proof. Let us consider the case $n = 2$ and let $z = \sum\limits_{i=1}^{\infty} a_i e_i$ is an arbitrary vector, such that $F_2(z) = 0$, i.e. $\sum\limits_{i=1}^{\infty} a_i^2 = 0$.

Let $z_0 = e_1 + c_1^{\frac{1}{2}} e_2 + c_2^{\frac{1}{2}} e_3 + c_3^{\frac{1}{2}} e_4 = (1, c_1^{\frac{1}{2}}, c_2^{\frac{1}{2}}, c_3^{\frac{1}{2}} 0, \ldots, 0) \in Z_k$. We will seek the vector z such that $F_2(z_0 + z) = 0$.

Let us choose $z = (a_1, a_2, a_3, a_4, 0, \ldots, 0)$ and $a_1, a_2, a_3, a_4 \neq 0$.

$$F_2(z_0 + z) = 2(a_1 + \sqrt{c_1} a_2 + \sqrt{c_2} a_3 + \sqrt{c_3} a_4).$$

Than we have equations:

$$\begin{cases} a_1^2 + a_2^2 + a_3^2 + a_4^2 = 0, \\ a_1 + \sqrt{c_1} a_2 + \sqrt{c_2} a_3 + \sqrt{c_3} a_4 = 0. \end{cases}$$

Hence

$$a_2 = \frac{-2(\sqrt{c_1 c_2} + a_4 \sqrt{c_1 c_3}) \pm 2(\sqrt{c_1} a_3 - \sqrt{c_2} a_4)}{2(c_1 + 1)} =$$

$$= \frac{a_3(-\sqrt{c_1 c_2} \pm \sqrt{c_3}) + a_4(-\sqrt{c_1 c_3} \pm \sqrt{c_2})}{c_1 + 1};$$

$$a_1 = -\sqrt{c_1} a_2 - \sqrt{c_2} a_3 - \sqrt{c_3} a_4 = \frac{a_3(c_1 \sqrt{c_2} \mp \sqrt{c_3 c_1}) + a_4(c_1 \sqrt{c_3} \mp \sqrt{c_2 c_1})}{c_1 + 1} -$$

$$-(\sqrt{c_2} a_3 + \sqrt{c_3} a_4) = \frac{a_3(-\sqrt{c_3} \mp \sqrt{c_1 c_3}) + a_4(-\sqrt{c_3} \mp \sqrt{c_2 c_1})}{c_1 + 1}.$$

Let us denote by

$$\beta_1 = \frac{a_3(-\sqrt{c_1 c_2} + \sqrt{c_3}) + a_4(-\sqrt{c_1 c_3} + \sqrt{c_2})}{c_1 + 1};$$

$$\alpha_1 = \frac{a_3(-\sqrt{c_3} - \sqrt{c_1c_3}) + a_4(-\sqrt{c_3} - \sqrt{c_2c_1})}{c_1+1};$$
$$\beta_2 = \frac{a_3(-\sqrt{c_1c_2} - \sqrt{c_3}) + a_4(-\sqrt{c_1c_3} - \sqrt{c_2})}{c_1+1};$$
$$\alpha_2 = \frac{a_3(-\sqrt{c_3} + \sqrt{c_1c_3}) + a_4(-\sqrt{c_3} + \sqrt{c_2c_1})}{c_1+1},$$

then we have got two different vectors of the form:

$$z_1 = (\alpha_1, \beta_1, a_3, a_4, 0, \ldots, 0)$$

and

$$z_2 = (\alpha_2, \beta_2, a_3, a_4, 0, \ldots, 0)$$

either of which are not belong to the space Z_k, i.e. the space Z_k is not maximal. □

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