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Solutions of Nonlinear PDE's of Fractional Order with Generalized Differential Transform Method

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Abstract

In this paper we use the generalized differential transform method for determining the approximate analytic solutions of fractional KdV, K(2,2) and mKdV equations.

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1 Introduction

Various phenomena in physics, like diffusion in a disordered or fractal medium, or in image analysis, or in risk management have been modeled by means of fractional partial differential equations. In general, there exists no method that yields an exact solution for these equations.

However, in the past decades, both mathematicians and physicists have devoted considerable effort to the study of explicit and numerical solutions to nonlinear differential equations of fractional order.

S. Momani and Z. Odibat have written a series of papers solving linear and nonlinear partial differential equations of fractional order (see [1-3]). Recently they developed a semi-numerical method for solving linear partial differential equations of fractional order [4]. This method is named as generalized differential transform method (GDTM) [5] and is based on the two-dimensional differential transform method (DTM) and generalized Taylors formula [6].

In the present paper we employ GDTM for solving some nonlinear partial differential equations with fractional order which already solved with homotopy analysis method (HAM) [7] and variational iteration method (VIM) for $\beta = 1$ [8] before.

2 Basic Definitions

In this section, we give some definitions and properties of the fractional calculus.

Definition 2.1 *A real function $f(t), t > 0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$, if there exists a real number $p(> \mu)$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C(0, \infty)$, and it is said to be in the space C_μ^n, if and only if $f^{(n)} \in C_\mu, n \in \mathbb{N}$.*

Definition 2.2 *The Riemann-Liouville fractional integral operator (J^α) of order $\alpha \geq 0$, of a function $f \in C_\mu$, $\mu \geq -1$, is defined as*

$$\begin{aligned} J^\alpha f(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad \alpha > 0, \\ J^0 f(t) &= f(t). \end{aligned}$$

Properties of the operator (J^α) can be found in [9,10], we mention only the following:

For $f \in C_\mu$, $\mu \geq -1$, $\quad \alpha, \beta \geq 0$ and $\gamma > -1$:

1. $J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t)$,

2. $J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t)$,

3. $J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$.

The Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D^α proposed by M. Caputo in his work on the theory of viscoelasticity [11].

Definition 2.3 *The fractional derivative of $f(t)$ in the Caputo sense is defined as*

$$D^\alpha f(t) = J^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds,$$

for $n - 1 < \alpha \leq n$, $n \in \mathbb{N}$, $t > 0$, $f \in C_{-1}^n$.

Also, we need here two of its basic properties.

Lemma 2.4 *If* $n-1<\alpha\leq n$, $n\in\mathbb{N}$ *and* $f\in C_\mu^n$, $\mu\geq -1$ *then*

$$(D^\alpha J^\alpha)f(t)=f(t)$$

and

$$(J^\alpha D^\alpha)f(t)=f(t)-\sum_{k=0}^{n-1} f^k(0^+)\frac{t^k}{k!},\quad t>0.$$

Definition 2.5 *For* n *to be the smallest integer that exceeds* α*, the Caputo time-fractional derivative operator of order* $\alpha>0$*, is defined as*

$$D_t^\alpha u(x,t)=\begin{cases}\frac{1}{\Gamma(n-\alpha)}\int_0^t\left[(t-s)^{n-\alpha-1}\frac{\partial^n}{\partial s^n}u(x,s)\right]ds, n-1<\alpha<n,\\ \frac{\partial^n u(x,t)}{\partial t^n}, \alpha=n\in\mathbb{N}.\end{cases}$$

For more information on the mathematical properties of fractional derivatives and integrals one can consult [12].

3 Analysis of the Method

Consider a function of two variables $u(x,t)$, and suppose that it can be represented as a product of two single-variable functions, i.e., $u(x,t)=f(x)g(t)$. Based on the properties of two-dimensional differential transform, the function $u(x,t)$ can be represented as

$$\begin{aligned} u(x,t) &= \sum_{k=0}^{\infty} F_\alpha(k)(x-x_0)^{k\alpha}\sum_{h=0}^{\infty} G_\beta(h)(t-t_0)^{h\beta} \\ &= \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U_{\alpha,\beta}(k,h)(x-x_0)^{k\alpha}(t-t_0)^{h\beta} \qquad (1)\end{aligned}$$

where $0<\alpha,\beta\leq 1$. If function $u(x,t)$ is analytic and differentiated continuously with respect to time t in the domain of interest, then we define the generalized two-dimensional differential transform of the function $u(x,t)$ as follows:

$$U_{\alpha,\beta}(k,h)=\frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}\left[(D_{x_0}^\alpha)^k(D_{t_0}^\beta)^h u(x,t)\right]_{(x_0,t_0)} \qquad (2)$$

where $(D_{x_0}^\alpha)^k=D_{x_0}^\alpha D_{x_0}^\alpha ... D_{x_0}^\alpha$, $k-$times. Besides, equation (1) is also called as the generalized inverse differential transform of $U_{\alpha,\beta}(k,h)$. In case of $\alpha=\beta=1$, then generalized two-dimensional differential transform (2) reduces to the classical two-dimensional differential transform.

Based on equations (1) and (2), we have the following properties [4].

1. If $u(x,t) = v(x,t) \pm w(x,t)$, then
$$U_{\alpha,\beta}(k,h) = V_{\alpha,\beta}(k,h) \pm W_{\alpha,\beta}(k,h).$$

2. If $u(x,t) = \lambda v(x,t)$, $\lambda \in R$, then
$$U_{\alpha,\beta}(k,h) = \lambda V_{\alpha,\beta}(k,h).$$

3. If $u(x,t) = v(x,t)w(x,t)$, then
$$U_{\alpha,\beta}(k,h) = \sum_{r=0}^{k}\sum_{s=0}^{h} V_{\alpha,\beta}(r,h-s)W_{\alpha,\beta}(k-r,s).$$

4. If $u(x,t) = (x-x_0)^{n\alpha}(t-t_0)^{m\beta}$, then
$$U_{\alpha,\beta}(k,h) = \delta\left(k-n\right)\delta\left(h-m\right).$$

5. If $u(x,t) = v(x,t)w(x,t)q(x,t)$, then
$$U_{\alpha,\beta}(k,h) = \sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s} V_{\alpha,\beta}(r,h-s-p)W_{\alpha,\beta}(t,s)Q_{\alpha,\beta}(k-r-t,p).$$

6. If $u(x,t) = f(x)g(t)$ and the function $f(x) = x^{\lambda}h(x)$, where $\lambda > -1$, $h(x)$ has the generalized Taylor series expansion $h(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^{k\alpha}$, and

   (a) $\beta < \lambda + 1$ and α arbitrary or

   (b) $\beta \geq \lambda + 1$, α arbitrary and $a_n = 0$ for $n = 0, 1, ..., m-1$, where $m-1 < \beta \leq m$.

   Then (2) becomes
$$U_{\alpha,\beta}(k,h) = \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}\left[D_{x_0}^{\alpha k}(D_{t_0}^{\beta})^{h}u(x,t)\right]_{(x_0,t_0)}.$$

7. If $u(x,t) = D_{x_0}^{\gamma}v(x,t), m-1 < \gamma \leq m$ and $v(x,t) = f(x)g(t)$ then
$$U_{\alpha,\beta}(k,h) = \frac{\Gamma(\alpha k+\gamma+1)}{\Gamma(\alpha k+1)}V_{\alpha,\beta}(k+\gamma/\alpha,h).$$

8. If $u(x,t) = D_{t_0}^{\gamma}v(x,t), m-1 < \gamma \leq m$ and $v(x,t) = f(x)g(t)$ then
$$U_{\alpha,\beta}(k,h) = \frac{\Gamma(\beta h+\gamma+1)}{\Gamma(\beta h+1)}V_{\alpha,\beta}(k,h+\gamma/\beta).$$

4 Test Problems

In this section, we present three examples in [7] to illustrate the applicability of GDTM to solve nonlinear fractional partial differential equations.

Example 4.1 *Consider the fractional KdV equation where* $0 < \beta \leq 1$

$$D_t^{\beta} u - 3(u^2)_x + u_{xxx} = 0, \tag{3}$$

with initial conditions $u(x,0) = 6x$, *[7].*
Taking the two-dimensional transform of (3) by using the related properties, we have

$$\begin{aligned} \frac{\Gamma(\beta(h+1)+1)}{\Gamma(\beta h+1)} U(k,h+1) &= 3(k+1)\sum_{r=0}^{k+1}\sum_{s=0}^{h} U(r,h-s)U(k-r+1,s) \\ &\quad -(k+1)(k+2)(k+3)U(k+3,h). \end{aligned} \tag{4}$$

The generalized two-dimensional transform of the initial condition can be obtained as follows:

$$\begin{aligned} U(k,0) &= 0, k = 0,2,3,\ldots \\ U(1,0) &= 6 \end{aligned} \tag{5}$$

By applying (5) into (4) we can obtain some value of $U(k,h)$ *as follows:*

$$\begin{aligned} U(k,1) &= 0, k = 0,2,3,\ldots \\ U(1,1) &= \frac{6^3}{\Gamma(\beta+1)} \\ U(k,2) &= 0, k = 0,2,3,\ldots \\ U(1,2) &= \frac{2.6^5}{\Gamma(2\beta+1)} \\ U(k,3) &= 0, k = 0,2,3,\ldots \\ U(1,3) &= \frac{4.6^7}{\Gamma(3\beta+1)} + \frac{6^7\Gamma(2\beta+1)}{\Gamma^2(\beta+1)\Gamma(3\beta+1)} \end{aligned}$$

Consequently substituting all $U(k,h)$ *into (1) and we obtain the series form solutions of (3) as*

$$\begin{aligned} u(x,t) &= 6x + \frac{6^3}{\Gamma(\beta+1)} x t^{\beta} + \frac{2.6^5}{\Gamma(2\beta+1)} x t^{2\beta} \\ &\quad + \left[\frac{4.6^7}{\Gamma(3\beta+1)} + \frac{6^7\Gamma(2\beta+1)}{\Gamma^2(\beta+1)\Gamma(3\beta+1)}\right] x t^{3\beta} + \ldots \end{aligned}$$

For the special case $\beta = 1$ *is [8]*

$$u(x,t) = 6x\left(1 + 36t + 36^2t^2 + 36^3t^3 + \ldots\right) = \frac{6x}{1-36t}.$$

Example 4.2 *Consider the following fractional K(2, 2) equation where* $0 < \beta \leq 1$

$$D_t^{\beta} u + (u^2)_x + (u^2)_{xxx} = 0, \tag{6}$$

with initial conditions $u(x,0) = x$, *[7]. Taking the two-dimensional transform of (6) by using the related properties, we have*

$$\frac{\Gamma(\beta(h+1)+1)}{\Gamma(\beta h+1)} U(k,h+1) = -(k+1) \sum_{r=0}^{k+1} \sum_{s=0}^{h} U(r,h-s) U(k-r+1,s)$$
$$-(k+1)(k+2)(k+3) \sum_{r=0}^{k+3} \sum_{s=0}^{h} U(r,h-s) U(k-r+3,s). \tag{7}$$

The generalized two-dimensional transform of the initial condition can be obtained as follows:

$$\begin{aligned} U(k,0) &= 0, k = 0,2,3,\ldots \\ U(1,0) &= 1 \end{aligned} \tag{8}$$

By applying (8) into (7) we can obtain some value of $U(k,h)$ *as follows:*

$$\begin{aligned} U(k,1) &= 0, k = 0,2,3,\ldots \\ U(1,1) &= \frac{-2}{\Gamma(\beta+1)} \\ U(k,2) &= 0, k = 0,2,3,\ldots \\ U(1,2) &= \frac{2^3}{\Gamma(2\beta+1)} \\ U(k,3) &= 0, k = 0,2,3,\ldots \\ U(1,3) &= -\frac{2^5}{\Gamma(3\beta+1)} - \frac{2^3\Gamma(2\beta+1)}{\Gamma^2(\beta+1)\Gamma(3\beta+1)} \end{aligned}$$

Consequently substituting all $U(k,h)$ *into (1) and we obtain the series form solutions of (6) as*

$$u(x,t) = x - \frac{2xt^{\beta}}{\Gamma(\beta+1)} + \frac{2^3 xt^{2\beta}}{\Gamma(2\beta+1)} - \left(\frac{2^5}{\Gamma(3\beta+1)} + \frac{2^3\Gamma(2\beta+1)}{\Gamma^2(\beta+1)\Gamma(3\beta+1)}\right) xt^{3\beta} + \ldots$$

For the special case $\beta = 1$ *is [8]*

$$u(x,t) = x - 2xt + 4xt^2 - 8xt^3 + \ldots = \frac{x}{1+2t}.$$

Example 4.3 *Consider the modified fractional KdV (mKdV) equation where* $0 < \beta \leq 1$

$$D_t^{\beta} u + \frac{1}{2}(u^2)_x - u_{xx} = 0, \tag{9}$$

with initial conditions $u(x,0) = x$, *[7]. Taking the two-dimensional transform of (9) by using the related properties, we have*

$$\begin{aligned} \frac{\Gamma(\beta(h+1)+1)}{\Gamma(\beta h+1)} U(k,h+1) &= -\frac{1}{2}(k+1)\sum_{r=0}^{k+1}\sum_{s=0}^{h} U(r,h-s)U(k-r+1,s) \\ &\quad + (k+1)(k+2)\,U(k+2,h) \end{aligned} \tag{10}$$

The generalized two-dimensional transform of the initial condition can be obtained as follows:

$$\begin{aligned} U(k,0) &= 0, k = 0,2,3,\ldots \\ U(1,0) &= 1 \end{aligned} \tag{11}$$

By applying (11) into (10) we can obtain some value of $U(k,h)$ *as follows:*

$$\begin{aligned} U(k,1) &= 0, k = 0,2,3,\ldots \\ U(1,1) &= \frac{-1}{\Gamma(\beta+1)} \\ U(k,2) &= 0, k = 0,2,3,\ldots \\ U(1,2) &= \frac{2}{\Gamma(2\beta+1)} \\ U(k,3) &= 0, k = 0,2,3,\ldots \\ U(1,3) &= -\frac{4}{\Gamma(3\beta+1)} - \frac{\Gamma(2\beta+1)}{\Gamma^2(\beta+1)\,\Gamma(3\beta+1)} \end{aligned}$$

Consequently substituting all $U(k,h)$ *into (1) and we obtain the series form solutions of (9) as*

$$u(x,t) = x - \frac{xt^{\beta}}{\Gamma(\beta+1)} + \frac{2xt^{2\beta}}{\Gamma(2\beta+1)} - \left(\frac{4}{\Gamma(3\beta+1)} + \frac{\Gamma(2\beta+1)}{\Gamma^2(\beta+1)\Gamma(3\beta+1)}\right) xt^{3\beta} + \ldots$$

For the special case $\beta = 1$ *is [8]*

$$u(x,t) = x - xt + xt^2 - xt^3 + \ldots = \frac{x}{1+t}.$$

5 Conclusion

For illustration purposes, we considered three examples. Results obtained using the scheme presented here agree well with the numerical results presented elsewhere. Results also show that the numerical scheme is very effective and convenient for solving nonlinear partial differential equations of fractional order. Numerical computations associated with the three examples discussed above were performed by using the Computer Algebra System MAPLE.

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