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Mean Square Solutions of Second-Order Random Differential Equations by Using Homotopy Perturbation Method

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Abstract

In this paper, the Homotopy perturbation method (HPM) is successfully applied for analytic (approximate) mean square solutions of the second-order random differential equations. Expectation and variance of the approximate solutions are computed. Several numerical examples are presented to show the ability and efficiency of this method .

Keywords: Random differential equations, Stochastic differential equation and Homotopy perturbation method

1. Introduction

A random ordinary differential equations are an ordinary differential equations which contains random constants or random variables. Most scientific problems, biology , engineering and physical phenomena occur in the form of random differential equations [1-3]. Recently, several first-order random differential models are solved using mean square calculus [4-11]. Many scientific models can be described as a second-order random differential equation in the following form

$$L[X(t)] + N[X(t), A] = g(t), \qquad (1)$$

$$X(0) = Y_0, \quad \frac{dX(t)}{dt}\Big|_{t=0} = Y_{1,} \qquad (2)$$

where $L[X(t)] = \dfrac{d^2 X(t)}{dt^2}$, $N[X(t), A]$ is a nonlinear operator and $g(t)$ is the source inhomogeneous term, as well as A, Y_0 and Y_1 are random variables. Within recent years, a special class of the initial value problem (1) has been treated under appropriate hypotheses on the data to evaluate the main statistical functions, such as the mean and the variance, of the approximate solution stochastic process generated by truncation of the exact power series solution[12-13]. More recently, variational iteration method and Adomian decomposition method are employed to find analytic (approximate) mean square solutions of the second-order random differential equations, homogeneous or inhomogeneous [14,15].

In this paper, the HPM is used to find the mean square solutions for second-order random initial value problems. Several numerical examples are implemented to show the efficiency of this method.

2. Homotopy Perturbation Method

The Homotopy Perturbation Method (HPM) was introduced by He [16,17]. The HPM is useful to obtain exact and approximate solutions of linear and nonlinear differential equations. To illustrate the basic ideas of HPM, let us consider the second-order random differential equation (1). By the homotopy technique, we construct a homotopy

$$H(X(t), p) = (1-p)\{L[X(t)] - L[x_0(t)]\} + p\{L[X(t)] + N[X(t), A] - g(t)\} = 0 \quad (3)$$

where $p \in [0,1]$ is an embedding parameter, $x_0(t)$ is an initial approximation of the equation (1) which satisfies the initial conditions (2). Obviously, from equation (3), we have

$$H(X(t), 0) = L[X(t)] - L[x_0(t)] = 0$$

$$H(X(t), 1) = L[X(t)] + N[X(t), A] - g(t) = 0$$

The changing process of p from 0 to 1 is just that of $X(t, p)$ from $x_0(t)$ to $X(t)$. In topology, According to HPM, we can first use the embedding parameter p as a small parameter, and assume that the solution of Eqs. (1) and (2) as a power series in p:

$$X(t, p) = x_0(t) + x_1(t)p + x_2(t)p^2 + x_3(t)p^3 + \cdots \quad (4)$$

Setting $p = 1$, results in the approximation to the solution of Eq.(1)

$$X(t) = \lim_{p \to 1} X(t, p) = x_0(t) + x_1(t) + x_2(t) + x_3(t) + \cdots \quad (5)$$

The combination of the perturbation method and the homotopy method is called the homotopy perturbation method (HPM), which has eliminated limitations of the traditional perturbation techniques.

3. Statistical Functions of the Mean Square Random HPM solution

This section concern with the computation of the main statistical functions of the m.s. solution of (1) given by the iteration formula (5).

$$E[X_N(t)] \approx \sum_{k=0}^{N} E[x_k(t)] \tag{6}$$

$$E[X_N^2(t)] \approx E[\left(\sum_{k=0}^{N} x_k(t)\right)^2] \tag{7}$$

$$V[X_N(t)] \approx E[\left(\sum_{k=0}^{N} x_k(t)\right)^2] - \left(\sum_{k=0}^{N} E[x_k(t)]\right)^2 \tag{8}$$

The following Lemma guarantee the convergent of the sequence $E[X_N(t)]$ to $E[X(t)]$ and the sequence $V[X_N(t)]$ to $V[X(t)]$ if the sequence the $X_N(t)$ converges to $X(t)$.

Lemma[5]: Let $\{X_N\}$ and $\{Y_N\}$ be two sequences of 2-r.vs X and Y, respectively , i.e. , $\lim_{N\to\infty} X_N = X$ and $\lim_{N\to\infty} Y_N = Y$ then $\lim_{N\to\infty} E[X_N Y_N] = E[XY]$

If $X_N = Y_N$,then $\lim_{N\to\infty} E[X_N^2] = E[X^2]$, $\lim_{N\to\infty} E[X_N] = E[X]$ and $\lim_{N\to\infty} V[X_N] = V[X]$.

4. Test Examples

In this section, we adopt several examples to illustrate the using of Homotopy perturbation method for approximating the mean and the variance. The results are computed by using Maple 14 and compared to exact solution .

Example 1 [13] : Consider random initial value problem $\frac{d^2X(t)}{dt^2} + A^2 X(t) = 0$, $X(0) = Y_0$ and $\left.\frac{dX(t)}{dt}\right|_{t=0} = Y_1$ where A^2 is a Beta r.v. with parameters $\alpha = 2$ and $\beta = 1$, i.e. $A^2 \sim Be(\alpha = 2, \beta = 1)$ and independently of the initial conditions Y_0 and Y_1 which satisfy $E[Y_o] = 1$, $E[Y_o^2] = 2$,

$E[Y_1]=1$, $E[Y_1^2]=3$ and $E[Y_0Y_1]=0$.

$$H(X(t),p)=(1-p)\{\frac{d^2X(t)}{dt^2}-\frac{d^2x_0(t)}{dt^2}\}+p\{\frac{d^2X(t)}{dt^2}+A^2X(t)\}=0 \qquad (9)$$

where $x_0(t)=Y_0+Y_1t$

Substituting (4) into (9), and some algebraic manipulations and rearranging the coefficients of the terms with identical powers of p, we have:

$$p^1:\ x_1''(t)+A^2x_0(t)=0 \quad \Rightarrow \quad x_1(t)=-\frac{1}{6}A^2Y_1t^3-\frac{1}{2}A^2Y_0t^2$$

$$p^2:\ x_2''(t)+A^2x_1(t)=0 \quad \Rightarrow \quad x_2(t)=\frac{1}{120}A^4Y_1t^5+\frac{1}{24}A^4Y_0t^4,$$

$$p^3:\ x_3''(t)+A^2x_2(t)=0 \quad \Rightarrow \quad x_3(t)=-\frac{1}{5040}A^6Y_1t^7-\frac{1}{720}A^6Y_0t^6$$

$$p^4:\ x_4''(t)+A^2x_3(t)=0 \quad \Rightarrow \quad x_4(t)=\frac{1}{362880}A^8Y_1t^9+\frac{1}{40320}A^8Y_0t^8$$

$$p^5:\ x_5''(t)+A^2x_4(t)=0 \quad \Rightarrow \quad x_5(t)=-\frac{1}{39916800}A^{10}Y_1t^{11}-\frac{1}{3628800}A^{10}Y_0t^{10}$$

$$\vdots$$

For N=5 , one can have

$$X_5(t)=Y_0+Y_1t-\frac{1}{6}A^2Y_1t^3-\frac{1}{2}A^2Y_0t^2+\frac{1}{120}A^4Y_1t^5+\frac{1}{24}A^4Y_0t^4-\frac{1}{5040}A^6Y_1t^7-\frac{1}{720}A^6Y_0t^6$$
$$+\frac{1}{362880}A^8Y_1t^9+\frac{1}{40320}A^8Y_0t^8-\frac{1}{39916800}A^{10}Y_1t^{11}-\frac{1}{3628800}A^{10}Y_0t^{10}$$

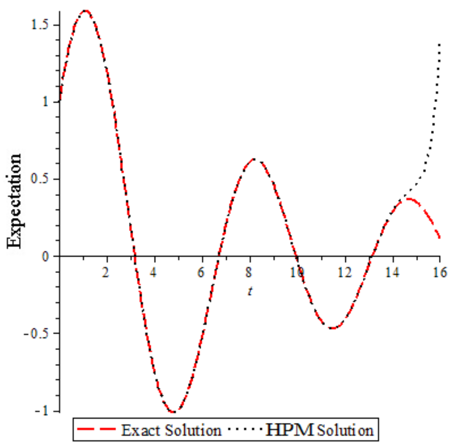

Fig.(1) : Comparison between the exact expectation and its approximation obtained from the HPM with N=18

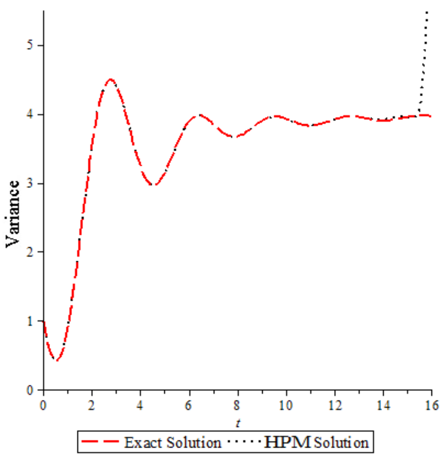

Fig.(2) : Comparison between the exact variance and its approximation obtained from the HPM with N=18

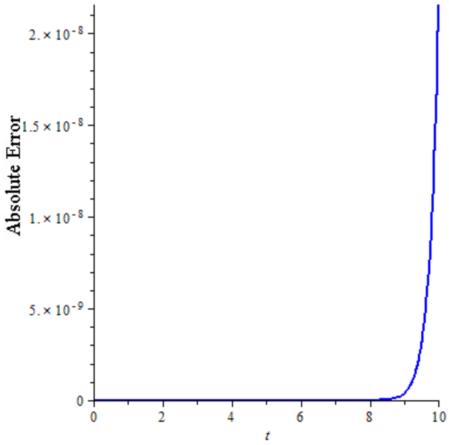

Fig.(3) : Absolute Error of expectation with N=18

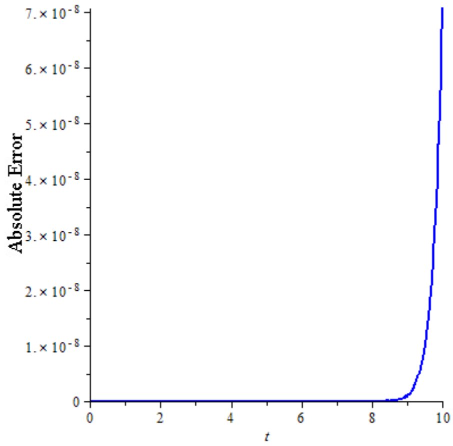

Fig.(4) : Absolute Error of variance with N=18

Example 2 [12] : Consider random initial value problem $\frac{d^2X(t)}{dt^2}+At\,X(t)=0$ $X(0)=Y_0$ and $\frac{dX(t)}{dt}\Big|_{t=0}=Y_1$ where A is a Beta r.v. with parameters $\alpha=2$ and $\beta=3$, i.e. $A \sim Be(\alpha=2,\beta=3)$ and independently of the initial conditions Y_o and Y_1 which are independent r.v.'s such as $E[Y_o]=1$, $E[Y_o^2]=2$, $E[Y_1]=2$, $E[Y_1{}^2]=5$.

$$H(X(t),p)=(1-p)\{\frac{d^2X(t)}{dt^2}-\frac{d^2x_0(t)}{dt^2}\}+p\{\frac{d^2X(t)}{dt^2}+At\,X(t)\}=0 \qquad (10)$$

where $x_0(t)=Y_0+Y_1t$

Substituting (4) into (10), and some algebraic manipulations and rearranging the coefficients of the terms with identical powers of p, we have:

$$p^1:\ x_1''(t)+At\,x_0(t)=0 \quad\Rightarrow\quad x_1(t)=-\frac{1}{12}AY_1t^4-\frac{1}{6}AY_0t^3\ ,$$

$$p^2:\ x_2''(t)+At\,x_1(t)=0 \quad\Rightarrow\quad x_2(t)=\frac{1}{504}A^2Y_1t^7+\frac{1}{180}A^2Y_0t^6,$$

$$p^3:\ x_3''(t)+At\,x_2(t)=0 \quad\Rightarrow\quad x_3(t)=-\frac{1}{45360}A^3Y_1t^{10}-\frac{1}{12960}A^3Y_0t^9,$$

$$p^4: \; x_4''(t) + At\,x_3(t) = 0 \qquad \Rightarrow \qquad x_4(t) = \frac{1}{7076160}A^4Y_1t^{13} + \frac{1}{1710720}A^4Y_0t^{12}$$

$$p^5: \; x_5''(t) + At\,x_4(t) = 0 \qquad \Rightarrow \qquad x_5(t) = -\frac{1}{1698278400}A^5Y_1t^{16} - \frac{1}{359251200}A^5Y_0t^{15}$$

$$\vdots$$

For N=5 , one can have

$$X_5(t) = Y_0 + Y_1t - \frac{1}{12}AY_1t^4 - \frac{1}{6}AY_0t^3 + \frac{1}{504}A^2Y_1t^7 + \frac{1}{180}A^2Y_0t^6 - \frac{1}{45360}A^3Y_1t^{10} - \frac{1}{12960}A^3Y_0t^9$$

$$+\frac{1}{7076160}A^4Y_1t^{13} + \frac{1}{1710720}A^4Y_0t^{12} - \frac{1}{1698278400}A^5Y_1t^{16} - \frac{1}{359251200}A^5Y_0t^{15}$$

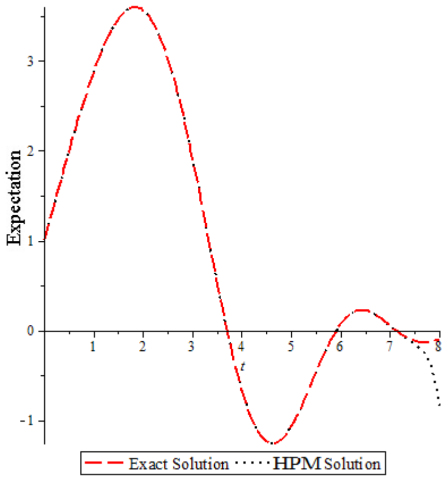

Fig.(5) : Comparison between the exact expectation and its approximation obtained from the HPM with N=15

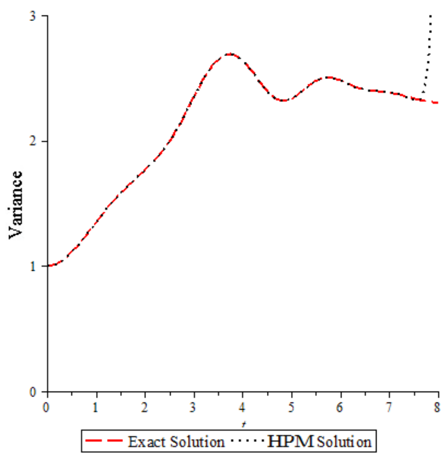

Fig.(6) : Comparison between the exact variance and its approximation obtained from the HPM with N=15

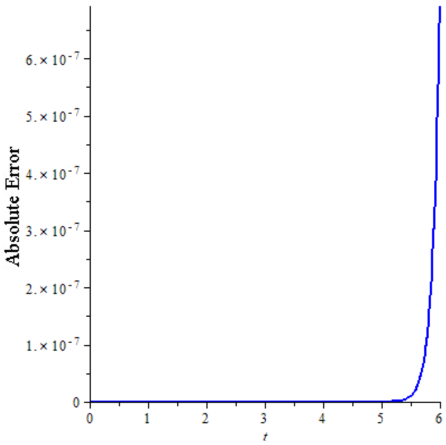

Fig.(7) : Absolute Error of expectation with N=15

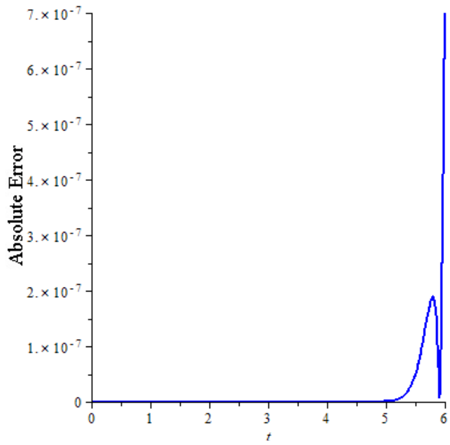

Fig.(8) : Absolute Error of variance with N=15

Example 3: Consider the problem $\frac{d^2X(t)}{dt^2}+2A\frac{dX(t)}{dt}+A^2\,X(t)=0$, $X\,(0)=Y_0$ and $\frac{dX\,(t)}{dt}\Big|_{t=0}=Y_1$ where A is a Beta r.v. with parameters $\alpha=2$ and $\beta=1$, i.e. $A \sim Be(\alpha=2,\beta=1)$ and independently of the initial conditions Y_o and Y_1 which are independent r.v.' satisfy $E[Y_o]=1$, $E[Y_o^2]=2$, $E[Y_1]=1$, $E[Y_1{}^2]=1$.

$$H(X(t),p)=(1-p)\{\frac{d^2X(t)}{dt^2}-\frac{d^2x_0(t)}{dt^2}\}+p\{\frac{d^2X(t)}{dt^2}+2A\frac{dX(t)}{dt}+A^2\,X(t)\}=0 \qquad (11)$$

where $x_0(t)=Y_0+Y_1t$

Substituting (4) into (11), and some algebraic manipulations and rearranging the coefficients of the terms with identical powers of p, we have:

$p^1:\ x_1''(t)+2A\,x_0'(t)+A^2\,x_0(t)=0\ \Rightarrow\quad x_1(t)=-\frac{1}{6}A^2Y_1t^3-t^2AY_1-\frac{1}{2}t^2A^2Y_0$

$p^2:\ x_2''(t)+2A\,x_1'(t)+A^2\,x_1(t)=0\ \Rightarrow$

$$x_2(t)=\frac{1}{120}A^4Y_1t^5+\frac{1}{6}t^4A^3Y_1+\frac{1}{24}t^4A^4Y_0+\frac{2}{3}A^2Y_1t^3+\frac{1}{3}A^3t^3Y_0$$

$p^3:\ x_3''(t)+2A\,x_2'(t)+A^2\,x_2(t)=0\ \Rightarrow$

$$x_3(t)=-\frac{1}{5040}A^6Y_1t^7-\frac{1}{120}t^6A^5Y_1-\frac{1}{720}t^6A^6Y_0-\frac{1}{10}A^4Y_1t^5-\frac{1}{30}t^5A^5Y_0-\frac{1}{6}t^4A^4Y_0-\frac{1}{3}t^4A^3Y_1$$

$$p^4: \; x_4''(t)+2A\, x_3'(t)+A^2\, x_3(t)=0 \;\Rightarrow$$

$$x_4(t)=\frac{1}{362880}A^8Y_1t^9+\frac{1}{5040}t^8A^7Y_1+\frac{1}{40320}t^8A^8Y_0+\frac{1}{210}A^6Y_1t^7+\frac{1}{840}t^7A^7Y_0+\frac{1}{60}t^6A^6Y_0$$
$$+\frac{2}{45}t^6A^5Y_1+\frac{1}{15}t^5A^5Y_0+\frac{2}{15}A^4Y_1t^5$$

$$p^5: \; x_5''(t)+2A\, x_4'(t)+A^2\, x_4(t)=0 \;\Rightarrow$$

$$x_5(t)=-\frac{1}{39916800}A^{10}Y_1t^{11}-\frac{1}{362880}t^{10}A^9Y_1-\frac{1}{3628800}t^{10}A^{10}Y_0-\frac{1}{9072}A^8Y_1t^9-\frac{1}{45360}t^9A^9Y_0$$
$$-\frac{1}{1680}t^8A^8Y_0-\frac{1}{504}t^8A^7Y_1-\frac{2}{315}t^7A^7Y_0-\frac{1}{63}A^6Y_1t^7-\frac{1}{45}t^6A^6Y_0-\frac{2}{45}t^6A^5Y_1$$
$$\vdots$$

For N=5 , one can have

$$X_5(t)=Y_0+Y_1t-\frac{1}{6}A^2Y_1t^3-t^2AY_1-\frac{1}{2}t^2A^2Y_0+\frac{1}{120}A^4Y_1t^5+\frac{1}{6}t^4A^3Y_1+\frac{1}{24}t^4A^4Y_0+\frac{2}{3}A^2Y_1t^3+\frac{1}{3}A^3t^3Y_0$$
$$-\frac{1}{5040}A^6Y_1t^7-\frac{1}{120}t^6A^5Y_1-\frac{1}{720}t^6A^6Y_0-\frac{1}{10}A^4Y_1t^5-\frac{1}{30}t^5A^5Y_0-\frac{1}{6}t^4A^4Y_0-\frac{1}{3}t^4A^3Y_1+\frac{1}{362880}A^8Y_1t^9$$
$$+\frac{1}{5040}t^8A^7Y_1+\frac{1}{40320}t^8A^8Y_0+\frac{1}{210}A^6Y_1t^7+\frac{1}{840}t^7A^7Y_0+\frac{1}{60}t^6A^6Y_0+\frac{2}{45}t^6A^5Y_1+\frac{1}{15}t^5A^5Y_0+\frac{2}{15}A^4Y_1t^5$$
$$-\frac{1}{39916800}A^{10}Y_1t^{11}-\frac{1}{362880}t^{10}A^9Y_1-\frac{1}{3628800}t^{10}A^{10}Y_0-\frac{1}{9072}A^8Y_1t^9-\frac{1}{45360}t^9A^9Y_0-\frac{1}{1680}t^8A^8Y_0$$
$$-\frac{1}{504}t^8A^7Y_1-\frac{2}{315}t^7A^7Y_0-\frac{1}{63}A^6Y_1t^7-\frac{1}{45}t^6A^6Y_0-\frac{2}{45}t^6A^5Y_1$$

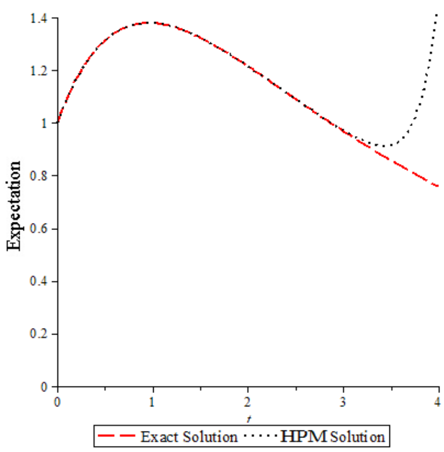

Fig.(9) : Comparison between the exact expectation and its approximation obtained from the HPM with N=16

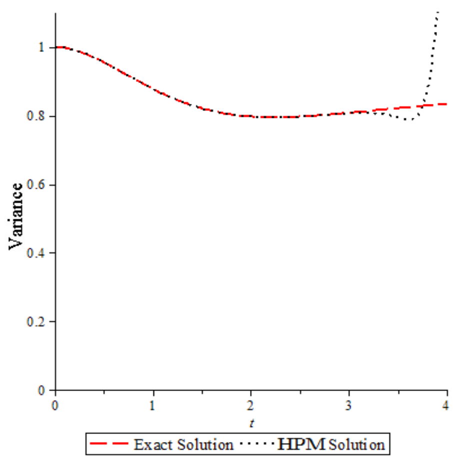

Fig.(10) : Comparison between the exact variance and its approximation obtained from the HPM with N=16

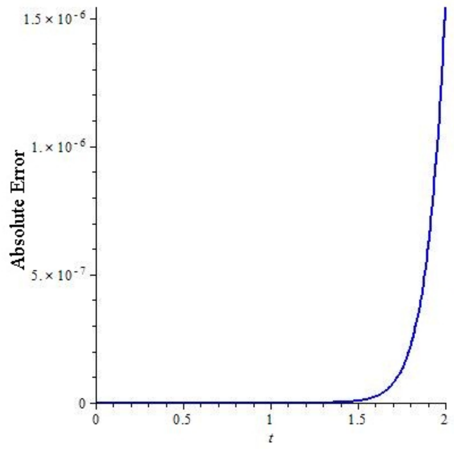

Fig.(11) : Absolute Error of expectation with N=16

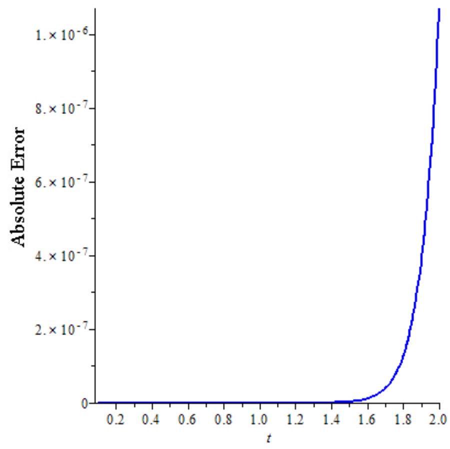

Fig.(12) : Absolute Error of variance with N=16

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