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Addendum to "A Simple Differential Equation System for the Description of Competition among Religions"

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Abstract

We propose a linear differential equation system for the description of competitions among populations (e.g. religions) for followers. The interaction among the populations is modeled by the use of constant coefficients and, as the new feature, by additional damping factors.

Mathematics Subject Classification: 91C99

Keywords: population dynamics, opinion formation, religions, conversion, adherence, differential equation system

1 Introduction

The growth and decline of religions in terms of their populations is a complex social phenomenon. A central aspect is the competition of coexisting religions for followers. Understood as quantitative objects, populations along with their increase and decrease can be expressed in mathematical terms, in particular in the form of differential equations. Thus, it is tempting to try a description of religious competition by a suitable set of differential equations. In fact, several models have been developed in the last years by researchers from various disciplines, see for example [1] (physics) or [2] (economics). Recently, we have proposed such a model in terms of differential equations [3] and we are fully aware of its simplicity compared to the complexity of the social reality. In the present paper we outline an alternative, in a sense even more simple model.

Consider a population of N individuals in total. How can the interactions among all these individuals and subsequent processes like group forming and group dissolution be modeled mathematically? A common approach is the use of an incidence matrix $\mathcal{N}$ with elements $n_{i,j}$ where $i, j = 1, \ldots, N$. Element $n_{i,j}$ indicates the relation between elements i and j, often one just has $n_{i,j} = 1$ or 0 indicating whether the individuals know each other or not. Although

$\mathcal{N}$ is a very powerful tool its use is hampered by the fast growing number of interactions $N(N-1)/2$. The opposite approach consists in the use of population numbers x_i with $\sum_i^R x_i = N$ and usually $R \ll N$. Interactions between populations are then expressed by differential equations which contain products like $x_i x_j$. The most famous example in population dynamics for this method is the prey-predator system. In [3] we outlined such a differential equation system for the competition of two religions and one irreligious group.

2 The differential equation system

In the present addendum to [3] we propose yet another approach. First of all, remember that the population x_i is a function $x_i(t)$ of time t. The interaction in time of population $x_i(t)$ with population $x_j(t)$ takes place via some (cultural) mechanism. As a consequence, $x_i(t)$ can gain followers from $x_j(t)$ according to some gain coefficient $\epsilon_{i,j}$ but this coefficient is damped by a damping factor $f(i)$. The damping factor (it could be defined as an amplification factor in an alternative set-up) describes in an overall manner the intensity of the mutual interaction among $x_i(t)$ and $x_j(t)$, its form is

$$f(i) = \mathrm{e}^{\frac{\zeta\,(x_i(t)-N(t))}{N(t)}} \tag{1}$$

with

$$N(t) = x_1(t) + x_2(t) +, \ldots, + x_R(t) \tag{2}$$

for up to R different religions. The coupling strength is expressed by the coupling factor $0 \le \zeta$. Note that is is always $0 \le f(i) \le 1$ since $x_i \le N(t)$. Define the term $T_0(i,j) = 0$ if $i = j$ and $T_0(i,j) = 1$ if $i \ne j$ and the opposite term $T_1(i,j) = 1$ if $i = j$ and $T_1(i,j) = 0$ if $i \ne j$. As in [3] we have terms α_i which represent the growth rate of population i, but for simplicity here we omit death rates. Then our new differential equation system DE is

$$\frac{d}{dt}x_i\,(t) = \sum_{i=1}^{R}\sum_{j=1}^{R}\left(\epsilon_{i,j}\,f(i)\,T_0(i,j) + \alpha_i\,T_1(i,j)\right)\,x_i(t) - \epsilon_{j,i}\,f(j)T_0(i,j)\,x_j(t) \tag{3}$$

Let us restrict to the case of $R = 3$ interacting groups, like in [3]. Then we can write the DE (3) in explicit form as

$$\begin{aligned}\frac{d}{dt}x_1\,(t) \;&=\; \epsilon_{1,2}\mathrm{e}^{\frac{\zeta\,(-x_2(t)-x_3(t))}{x_1(t)+x_2(t)+x_3(t)}}x_1\,(t) - \epsilon_{2,1}\mathrm{e}^{\frac{\zeta\,(-x_1(t)-x_3(t))}{x_1(t)+x_2(t)+x_3(t)}}x_2\,(t) \\ &+\epsilon_{1,3}\mathrm{e}^{\frac{\zeta\,(-x_2(t)-x_3(t))}{x_1(t)+x_2(t)+x_3(t)}}x_1\,(t) - \epsilon_{3,1}\mathrm{e}^{\frac{\zeta\,(-x_1(t)-x_2(t))}{x_1(t)+x_2(t)+x_3(t)}}x_3\,(t) + \alpha_1 x_1\,(t)\end{aligned} \tag{4}$$

$$\frac{d}{dt}x_2(t) = \epsilon_{2,1}e^{\frac{\zeta(-x_1(t)-x_3(t))}{x_1(t)+x_2(t)+x_3(t)}}x_2(t) - \epsilon_{1,2}e^{\frac{\zeta(-x_2(t)-x_3(t))}{x_1(t)+x_2(t)+x_3(t)}}x_1(t) + \epsilon_{2,3}e^{\frac{\zeta(-x_1(t)-x_3(t))}{x_1(t)+x_2(t)+x_3(t)}}x_2(t) - \epsilon_{3,2}e^{\frac{\zeta(-x_1(t)-x_2(t))}{x_1(t)+x_2(t)+x_3(t)}}x_3(t) + \alpha_2 x_2(t) \quad (5)$$

$$\frac{d}{dt}x_3(t) = \epsilon_{3,1}e^{\frac{\zeta(-x_1(t)-x_2(t))}{x_1(t)+x_2(t)+x_3(t)}}x_3(t) - \epsilon_{1,3}e^{\frac{\zeta(-x_2(t)-x_3(t))}{x_1(t)+x_2(t)+x_3(t)}}x_1(t) + \epsilon_{3,2}e^{\frac{\zeta(-x_1(t)-x_2(t))}{x_1(t)+x_2(t)+x_3(t)}}x_3(t) - \epsilon_{2,3}e^{\frac{\zeta(-x_1(t)-x_3(t))}{x_1(t)+x_2(t)+x_3(t)}}x_2(t) + \alpha_3 x_3(t) \quad (6)$$

3 Numerical solutions

The evolution of $x_i(t)$ with time t can be calculated numerically from 4, 5, 6. As in [3] we abet R_3 with the greatest growth rate: $\alpha_1 = 0.001$, $\alpha_2 = 0.001$, $\alpha_3 = 0.02$. Furthermore both R_2 and R_3 gain followers from R_1, R_3 has an additional inflow from R_2, R_1 has no inflow: $\epsilon_{2,1} = 0.01$, $\epsilon_{3,1} = 0.005$, $\epsilon_{3,2} = 0.01$. These coefficients are chosen arbitrarily for our present examples, they need to be fitted to empirical data for empirical studies. In the present examples we just concentrate on the role of the coupling factor ζ.

Figure 1 exhibits the case $\zeta = 0.0$ for which no damping of the interaction occurs. One observes the rapid decline of R_1 and contrarily the rise of R_3. Figure 2 shows the case $\zeta = 1.0$ with strong damping of the interaction. In that case, R_1 survives.

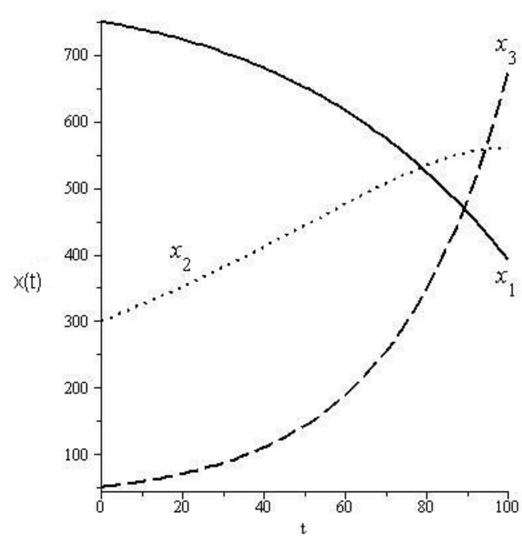

Figure 1: Solution to (4) - (6) for $\zeta = 0.0$.

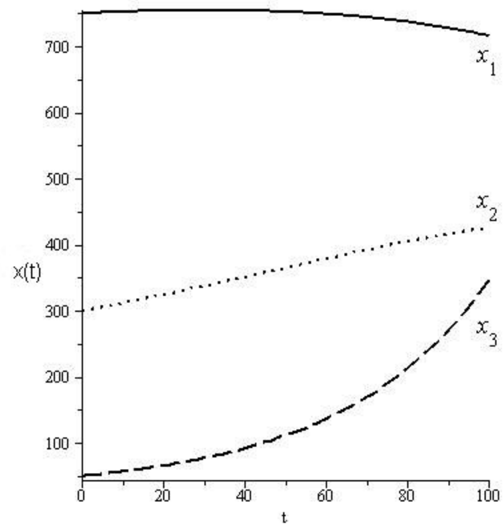

Figure 2: Solution to (4) - (6) $\zeta = 1.0$.

4 Discussion and Conclusion

The present approach to the growth and decline of religions in terms of their populations introduces the damping factors $f(i)$ as a new element into the modeling by differential equations. These $f(i)$ could help to describe (in a very broad manner) for example the dependency of the interactions on the local distributions of the populations. The actual form of the $f(i)$ may be adapted to the concrete problem under consideration. In its proposed form, the equation system (3) is linear in the x_i which is at least more easy to treat then a non-linear system with products like $x_i x_j$. Anyhow, further progress in the modeling will depend on the incorporation of empirical data.

5 Appendix

The following commands will set up the DE (4), (5), (6) within the computer algebra system Maple (version 13).

```
> restart:
> with(plots): with(DEtools):
> T0 := proc (a, b) local Result; if a = b then Result := 0
else Result := 1 end if; Result end proc;
T1 := proc (a, b) local Result; if a = b then Result := 1
else Result := 0 end if; Result end proc;
> iend := 3; jend := iend;
> NA := add(x[dummy](t), dummy = 1 .. jend);
> i := 'i'; j := 'j'; epsilon := 'epsilon';
```

```
alpha := 'alpha'; zeta := 'zeta';
envti := exp(zeta*(x[i](t)-NA)/NA);
envtj := exp(zeta*(x[j](t)-NA)/NA);
> for i to iend do DE[i] := diff(x[i](t), t) =
add((epsilon[i, j]*envti*T0(i, j)
+alpha[i]*T1(i, j))*x[i](t)
-epsilon[j, i]*envtj*T0(i, j)*x[j](t), j = 1 .. jend) end do;
```

The following Maple commands will solve the DE (4), (5), (6) in numerical manner.

```
> x1t0 := 750; x2t0 := 300; x3t0 := 50;
> for i to iend do for j to jend do
epsilon[i, j] := 0.1e-2 end do end do;
> epsilon[2, 1] := 0.1e-1; epsilon[3, 1] := 0.5e-2;
epsilon[3, 2] := 0.1e-1;
> for i to iend do dlt[i] := 0 end do;
> alpha[1] := 0.1e-2; alpha[2] := 0.1e-2;
alpha[3] := 0.2e-1;
> zeta := 1.0;
> ABPsys := [DE[1], DE[2], DE[3]];
> label1 := textplot([100, 270, 'x[1]', font = [TIMES, 15]]);
label2 := textplot([100, 460, 'x[2]', font = [TIMES, 15]]);
label3 := textplot([100, 680, 'x[3]'], font = [TIMES, 15]);
aplot:=DEplot(ABPsys, [x[1](t),x[2](t),x[3](t)], t = 0..100,
[[x[1](0)=x1t0,x[2](0)=x2t0,x[3](0)=x3t0]], scene=[t,x[1](t)],
thickness=2, linestyle=1, linecolor=black, stepsize=1.0):
bplot:=DEplot(ABPsys, [x[1](t),x[2](t),x[3](t)], t = 0..100,
[[x[1](0)=x1t0,x[2](0)=x2t0,x[3](0)=x3t0]], scene=[t,x[2](t)],
thickness=2, linestyle=2, linecolor=black, stepsize=1.0):
pplot:=DEplot(ABPsys, [x[1](t),x[2](t),x[3](t)], t = 0..100,
[[x[1](0)=x1t0,x[2](0)=x2t0,x[3](0)=x3t0]], scene=[t,x[3](t)],
thickness=2, linestyle=3, linecolor=black, stepsize=1.0):
display([aplot,bplot,pplot,label1,label2,label3]);
```

References

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